

Affine techniques on extremal metrics on toric surfaces

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Abstract

This paper consists of real and complex affine techniques for studying the Abreu equation on toric surfaces. In particular, an interior estimate for Ricci tensor is given.

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Studying the extremal metrics on compact toric manifolds has been an interesting project in the complex geometry in the past decade. The project was mainly formulated by Donaldson in [23]. Since then, some important progresses have been made. In particular, Donaldson([24]) completes the study of the metrics of cscK (abbreviated for constant scalar curvature) on toric surfaces. However, finding the extremal metrics on toric surfaces is still open. In this paper and its sequence([9]), we would study the metrics on toric surfaces with prescribed "scalar curvature" function K on the Delzant polytopes. In [9], we prove that, for any smooth function K on the Delzant polytope with a mild assumption (cf. Definition 1.6) and satisfying certain stability condition, there exists a Kähler metric on the toric surface whose "scalar curvature" is K . In particular, this includes most of extremal metrics.

This paper is completely a technique paper which is a preparation for [9]. The purpose of the paper is to explain the various techniques and estimates we develop for the Abreu equation. Since most of techniques are motivated by the studies of affine geometry by the second authors and his collaborators, we call the techniques to be the affine techniques.

As we know, though the equation of extremal metric is on the complex manifolds, for the toric manifolds, the equation can be reduced to a real equation on the Delzant polytope in real space. The second author with his collaborators develops a framework to study one type of 4-th order PDE's which includes the Abreu equation(cf. [10, 11, 34, 36–39, 42–44, 47]). We call this the *real affine technique*. This is explained through §2-§4. The whole package includes the differential inequality of Φ (cf. §2.3), the convergence theorem (cf. Theorem 2.8 and Theorem 3.6, §3.2), the Bernstein properties (cf. §3.3) and the affine blow-up analysis (cf. §4). This technique is very useful to study the equation in the interior of the polytope, for example, Theorem 4.2. The Theorem 2.8, Theorem 3.6 and Theorem 4.2 are also important for estimates on the boundary.

The challenging part of solving the extremal metric is to study the boundary behavior of the Abreu equation near the boundary of polytope. For the boundary of polytopes, they can be thought as the interior of the complex manifold. The challenging issue is then to generalize the affine techniques to the complex case. In this paper, we make an attempt on this direction. The results we develop are enough for us to study the Abreu equations and will be used in [9]. We call the technique we develop through §5-§7 the *complex affine technique*. We develop the differential inequalities for affine-type invariants in §5; as applications of these differential inequalities, we establish the interior estimates in §6; then in §7, we develop a convergence theorem (Theorem 7.15). With the aid of blow-up analysis, we are able to establish the estimate of the norm \mathcal{K} of the Ricci tensor ((1.3)) near the boundary (cf. Theorem 7.1). Theorem 7.1 is one of the most important theorem we need in [9].

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1 Kähler geometry on toric surfaces

We review the Kähler geometry over toric surfaces. We assume that the readers are familiar with toric varieties. The purpose of this section is to introduce the notations used in this paper.

A toric manifold is a $2n$ -dimensional Kähler manifold (M, ω) that admits an n -torus \mathbb{T}^n Hamiltonian action. Let $\tau : M \rightarrow \mathfrak{t}^*$ be the moment map, here $\mathfrak{t} \cong \mathbb{R}^n$ is the Lie algebra of \mathbb{T}^n and \mathfrak{t}^* is its dual. The image $\Delta = \tau(M)$ is known to be a polytope ([22]). In the literature, people use Δ for the image of the moment map. However, to be convenient in this paper, we always assume that Δ is an open polytope. Δ determines a fan Σ in \mathfrak{t} . The converse is not true: Σ determines Δ only up to certain similarity. M can be reconstructed from either Δ or Σ (cf. Chapter 1 in [27] and [29]). Moreover, the class of ω can be read from Δ . Hence, the uncertainty of Δ reflects the non-uniqueness of Kähler classes. Two different constructions are related via Legendre transformations. The Kähler geometry appears naturally when considering the transformation between two different constructions. This was explored by Guillemin ([29]). We will summarize these facts in this section. For simplicity, we only consider the toric surfaces, i.e, $n = 2$.

1.1 Toric surfaces and coordinate charts

Let Σ and Δ be a pair of fan and polytope for a toric surface M . For simplicity, we focus on compact toric surfaces. Then Δ is a Delzant polytope in \mathfrak{t}^* and its closure is compact.

We use the notations in §2.5([27]) to describe the fan. Let Σ be a fan given by a sequence of lattice points

$$\{v_0, v_1, \dots, v_{d-1}, v_d = v_0\}$$

in counterclockwise order, in $N = \mathbb{Z}^2 \subset \mathfrak{t}$ such that successive pairs generate the N .

Suppose that the vertices and edges of Δ are denoted by

$$\{\vartheta_0, \dots, \vartheta_d = \vartheta_0\}, \quad \{\ell_0, \ell_1, \dots, \ell_{d-1}, \ell_d = \ell_0\}.$$

Here $\vartheta_i = \ell_i \cap \ell_{i+1}$.

By saying that Σ is dual to Δ we mean that v_i is the inward pointing normal vector to ℓ_i of Δ . Hence, Σ is determined by Δ . Suppose that the equation for ℓ_i is

$$l_i(\xi) := \langle \xi, v_i \rangle - \lambda_i = 0. \quad (1.1)$$

Then

$$\Delta = \{\xi | l_i(\xi) > 0, \quad 0 \leq i \leq d-1\}$$

There are three types of cones in Σ : a 0-dimensional cone $\{0\}$ that is dual to Δ ; each of 1-dimensional cones generated by v_i that is dual to ℓ_i ; each of 2-dimensional cones generated by $\{v_i, v_{i+1}\}$ that is dual to ϑ_i . We denote them by Cone_Δ , Cone_{ℓ_i} and $\text{Cone}_{\vartheta_i}$ respectively.

It is known that for each cone of Σ , one can associate it a complex coordinate chart of M (cf. §1.3 and §1.4 in [27]). Let U_Δ, U_{ℓ_i} and U_{ϑ_i} be the coordinate charts. Then

$$U_\Delta \cong (\mathbb{C}^*)^2; \quad U_{\ell_i} \cong \mathbb{C} \times \mathbb{C}^*; \quad U_{\vartheta_i} \cong \mathbb{C}^2.$$

In particular, in each U_{ℓ_i} there is a divisor $\{0\} \times \mathbb{C}^*$. Its closure is a divisor in M , we denote it by Z_{ℓ_i} .

Remark 1.1 \mathbb{C}^* is called a complex torus and denoted by \mathbb{T}^c . Let z be its natural coordinate. $\mathbb{T} \cong S^1$ is called a real torus.

In this paper, we introduce another complex coordinate by considering the following identification

$$\mathbb{T}^c \rightarrow \mathbb{R} \times 2\sqrt{-1}\mathbb{T}; \quad w = \log z^2. \quad (1.2)$$

We call $w = x + 2\sqrt{-1}y$ the log-affine complex coordinate (or log-affine coordinate) of \mathbb{C}^* .

When $n = 2$, we have

$$(\mathbb{C}^*)^2 \cong \mathfrak{t} \times 2\sqrt{-1}\mathbb{T}^2.$$

Then (z_1, z_2) on the left hand side is the usual complex coordinate; while (w_1, w_2) on the right hand side is the log-affine coordinate. Write $w_i = x_i + 2\sqrt{-1}y_i$. Then (x_1, x_2) is the coordinate of \mathfrak{t} .

We make the following convention.

Remark 1.2 *On different type of coordinate charts, we use different coordinate systems:*

- on $U_\vartheta \cong \mathbb{C}^2$, we use the coordinate (z_1, z_2) ;
- on $U_\ell \cong \mathbb{C} \times \mathbb{C}^*$, we use the coordinate (z_1, w_2) ;
- on $U_\Delta \cong (\mathbb{C}^*)^2$, we use the coordinate (w_1, w_2) , or (z_1, z_2) , where $z_i = e^{\frac{w_i}{2}}, i = 1, 2$.

Remark 1.3 *Since we study the \mathbb{T}^2 -invariant geometry on M , it is useful to name a representative point of each \mathbb{T}^2 -orbit. Hence for $(\mathbb{C}^*)^2$, we let the points on $\mathfrak{t} \times 2\sqrt{-1}\{1\}$ to be the representative points.*

1.2 Kähler geometry on toric surfaces

One can read the class of symplectic form of M from Δ . In fact, Guillemin constructed a natural Kähler form ω_o and we denote the class by $[\omega_o]$ and we treat it as a canonical point in the class. We call it the Guillemin metric.

For each T^2 -invariant Kähler form $\omega \in [\omega_o]$, on each coordinate chart, there is a potential function (up to linear functions). Let

$$\mathbf{g} = \{(g_\Delta, g_{\ell_i}, g_{\vartheta_i}) | 0 \leq i \leq d-1\}$$

be the collection of potential functions on U_Δ, U_{ℓ_i} and U_{ϑ_i} for ω_o . For simplicity, we set $g = g_\Delta$.

Let $C_{\mathbb{T}^2}^\infty(M)$ be the smooth \mathbb{T}^2 -invariant functions of M . Set

$$C^\infty(M, \omega_o) = \{\mathbf{f} | \mathbf{f} = \mathbf{g} + \phi, \phi \in C_{\mathbb{T}^2}^\infty(M) \text{ and } \omega_f > 0\}.$$

Here

$$\mathbf{f} = \{(f_\Delta, f_{\ell_i}, f_{\vartheta_i}) | f_\bullet = g_\bullet + \phi\} \text{ and } \omega_f = \omega_o + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi.$$

In the definition, \bullet represents any of $\Delta, \ell_i, \vartheta_i$.

Remark 1.4 *Let $\phi \in C_{\mathbb{T}^2}^\infty(M)$ and g be any of $g_\Delta, g_{\ell_i}, g_{\vartheta_i}$. Set $f = g + \phi$. Consider the matrix*

$$\mathfrak{M}_f = \left(\sum_k g^{i\bar{k}} f_{j\bar{k}} \right).$$

Though this is not a globally well defined matrix on M , its eigenvalues are. Set ν_f to be the set of eigenvalues and $H_f = \det \mathfrak{M}_f^{-1}$. These are global functions on M .

Within a coordinate chart with potential function f , the Christoffel symbols, the curvature tensors, the Ricci curvature and the scalar curvature of Kähler metric ω_f are given by

$$\Gamma_{ij}^k = \sum_{l=1}^n f^{k\bar{l}} \frac{\partial f_{i\bar{l}}}{\partial z_j}, \quad \Gamma_{i\bar{j}}^{\bar{k}} = \sum_{l=1}^n f^{\bar{k}l} \frac{\partial f_{i\bar{l}}}{\partial z_{\bar{j}}},$$

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 f_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + \sum f^{p\bar{q}} \frac{\partial f_{i\bar{q}}}{\partial z_k} \frac{\partial f_{p\bar{j}}}{\partial \bar{z}_l},$$

$$R_{i\bar{j}} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} (\log \det (f_{k\bar{l}})), \quad \mathcal{S} = \sum f^{i\bar{j}} R_{i\bar{j}}.$$

respectively. When we use the log-affine coordinates and restricts on \mathfrak{t} ,

$$R_{i\bar{j}} = -\frac{\partial^2}{\partial x_i \partial x_j} (\log \det (f_{kl})), \quad \mathcal{S} = -\sum f^{ij} \frac{\partial^2}{\partial x_i \partial x_j} (\log \det (f_{ij})).$$

We treat \mathcal{S} as an operator of scalar curvature of f and denote it by $\mathcal{S}(f)$. Define

$$\mathcal{K} = \|Ric\|_f + \|\nabla Ric\|_f^{\frac{2}{3}} + \|\nabla^2 Ric\|_f^{\frac{1}{2}} \quad (1.3)$$

On the other hand, we denote by $\dot{\Gamma}_{ij}^k$, \dot{R}_{kil}^m and $\dot{R}_{i\bar{j}}$ the connections, the curvatures and the Ricci curvature of the metric ω_o respectively. They have similar formulas as above.

When focusing on \mathbb{U}_Δ and using the log-affine coordinate (cf. Remark 1.1), we have $f(x) = g(x) + \phi(x)$. We find that when restricting on $\mathbb{R}^2 \cong \mathbb{R}^2 \times 2\sqrt{-1}\{1\}$, the Riemannian metric induced from ω_f is the Calabi metric G_f (cf. §2.1).

We fix a large constant $K_o > 0$. Set

$$\mathcal{C}^\infty(M, \omega_o; K_o) = \{f \in \mathcal{C}^\infty(M, \omega_o) | \mathcal{S}(f) \leq K_o\}.$$

1.3 The Legendre transformation, moment maps and potential functions

Let f be a (smooth) strictly convex function on \mathfrak{t} . The gradient of f defines a (normal) map ∇^f from \mathfrak{t} to \mathfrak{t}^* :

$$\xi = (\xi_1, \xi_2) = \nabla^f(x) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \right).$$

The function u on \mathfrak{t}^*

$$u(\xi) = x \cdot \xi - f(x).$$

is called the Legendre transformation of f . We write $u = L(f)$. Conversely, $f = L(u)$.

Now we restrict on \mathbb{U}_Δ . When we choose the coordinate (z_1, z_2) , the moment map with respect to ω_f is given by

$$\tau_f : \mathbb{U}_\Delta \xrightarrow{(\log |z_1^2|, \log |z_2^2|)} \mathfrak{t} \xrightarrow{\nabla^f} \Delta \quad (1.4)$$

Note that the first map coincides with (1.2). Let $u = L(f)$. It is known that u must satisfy certain behavior near boundary of Δ . In fact, let $v = L(g)$, where g is the potential function of the Guillemin metric, then

Theorem 1.5 (Guillemin) $v(\xi) = \sum_i l_i \log l_i$, where l_i is defined in (1.1).

Then $u = v + \psi$, where $\psi \in C^\infty(\bar{\Delta})$. Set

$$C^\infty(\Delta, v) = \{u | u = v + \psi \text{ is strictly convex}, \psi \in C^\infty(\bar{\Delta})\}.$$

This space only depends on Δ and we treat v as a canonical point of the space.

We summarize the fact we just present: let $f \in C^\infty(M, \omega_o)$, then the moment map τ_f is given by $f = f_\Delta$ via the diagram (1.4) and $u = L(f) \in C^\infty(\Delta, v)$. In fact, the reverse is also true. This is explained in the following.

Given a function $u \in C^\infty(\Delta, v)$, we can get an $f \in C^\infty(M, \omega_o)$ as the following.

- On U_Δ , $f_\Delta = L(u)$;
- on U_ϑ , f_ϑ is constructed in the following steps: (i), suppose that ϑ is the intersection of two edges ℓ_1 and ℓ_2 , we choose a coordinate system (ξ_1, ξ_2) on t^* such that

$$\ell_i = \{\xi | \xi_i = 0\}, i = 1, 2, \quad \Delta \subset \{\xi | \xi_i \geq 0, i = 1, 2\};$$

then u is transformed to be a function in the following format

$$u' = \xi_1 \log \xi_1 + \xi_2 \log \xi_2 + \psi';$$

(ii), $f' = L(u')$ defines a function on t and therefore is a function on $(\mathbb{C}^*)^2 \subset U_\vartheta$ in terms of log-affine coordinate; (iii), it is known that f' can be extended over U_ϑ and we set f_ϑ to be this function;

- on U_ℓ , the construction of f_ℓ is similar to f_ϑ . Reader may refer to §1.5 for the construction.

We remark that u and $f = f_\Delta$ are determined by each other. Therefore, all f_ϑ , f_ℓ can be constructed from f . From this point of view, we may write f for f .

1.4 The Abreu equation on Δ

We can transform the scalar curvature operator $\mathcal{S}(f)$ to an operator $\mathcal{S}(u)$ of u on Δ . Then the equation $\mathcal{S}(f) = \mathcal{S}$ is transformed to be

$$\mathcal{S}(u) = \mathcal{S} \circ \nabla^u.$$

The operator $\mathcal{S}(u)$ is known to be

$$\mathcal{S}(u) = - \sum U^{ij} w_{ij}$$

where (U^{ij}) is the cofactor matrix of the Hessian matrix (u_{ij}) , $w = (\det(u_{ij}))^{-1}$. It is well known that ω_f gives an extremal metric if and only if $\mathcal{S} \circ \nabla^u$ is a linear function of Δ . Let K be a smooth function on $\bar{\Delta}$, the Abreu equation is

$$\mathcal{S}(u) = K. \tag{1.5}$$

We set $\mathcal{C}^\infty(\Delta, v; K_o)$ to be the functions in $u \in C^\infty(\Delta, v)$ with $\mathcal{S}(u) \leq K_o$.

Definition 1.6 *Let K be a smooth function on $\bar{\Delta}$. It is called edge-nonvanishing if it does not vanish on any edge of Δ .*

In our papers, we will always assume that K is edge-nonvanishing.

1.5 A special case: $\mathbb{C} \times \mathbb{C}^*$

Let $\mathfrak{h}^* \subset \mathfrak{t}^*$ be the half plane given by $\xi_1 \geq 0$. The boundary is ξ_2 -axis and we denote it by \mathfrak{t}_2^* . The corresponding fan consists of only one lattice $v = (1, 0)$. The coordinate chart is $\mathbb{U}_{\mathfrak{h}^*} = \mathbb{C} \times \mathbb{C}^*$. Let $Z = Z_{\mathfrak{t}_2^*} = \{0\} \times \mathbb{C}^*$ be its divisor.

Let $v_{\mathfrak{h}^*} = \xi_1 \log \xi_1 + \xi_2^2$. Set

$$\mathcal{C}^\infty(\mathfrak{h}^*, v_{\mathfrak{h}^*}) = \{u | u = v_{\mathfrak{h}^*} + \psi \text{ is strictly convex}, \psi \in C^\infty(\mathfrak{h}^*)\}$$

and $\mathcal{C}^\infty(\mathfrak{h}^*, v_{\mathfrak{h}^*}; K_o)$ be the functions whose \mathcal{S} is less than K_o .

Take a function $u \in \mathcal{C}^\infty(\mathfrak{h}^*, v_{\mathfrak{h}^*})$. Then $f = L(u)$ is a function on \mathfrak{t} . Hence it defines a function on the $\mathbb{C}^* \times \mathbb{C}^* \subset \mathbb{U}_{\mathfrak{h}^*}$ in terms of log-affine coordinate (w_1, w_2) . Then the function $f_{\mathfrak{h}}(z_1, w_2) := f(\log |z_1^2|, \text{Re}(w_2))$ extends smoothly over Z , hence defines on $\mathbb{U}_{\mathfrak{h}^*}$. We conclude that for any $u \in \mathcal{C}^\infty(\mathfrak{h}^*, v_{\mathfrak{h}^*})$ it yields a potential function $f_{\mathfrak{h}}$ on $\mathbb{U}_{\mathfrak{h}^*}$.

When we choose the coordinate (z_1, w_2) , the moment map with respect to ω_f is given by

$$\tau_f : \mathbb{U}_{\mathfrak{h}}^* \xrightarrow{(\log |z_1^2|, \text{Re}(w_2))} \mathfrak{t} \xrightarrow{\nabla^f} \mathfrak{h}^*. \quad (1.6)$$

Using $v_{\mathfrak{h}^*}$ and the above argument, we define a function $g_{\mathfrak{h}}$ on $\mathbb{U}_{\mathfrak{h}}^*$.

2 Calabi geometry

2.1 Calabi metrics and basic affine invariants

Let $f(x)$ be a smooth, strictly convex function defined on a convex domain $\Omega \subset \mathbb{R}^n \cong \mathfrak{t}$. As f is strictly convex,

$$G := G_f = \sum_{i,j} f_{ij} dx_i dx_j$$

defines a Riemannian metric on Ω . We call it the *Calabi* metric. We recall some fundamental facts on the Riemannian manifold (Ω, G) (cf. [46]). The Levi-Civita connection is given by

$$\Gamma_{ij}^k = \frac{1}{2} \sum f^{kl} f_{ijl}.$$

The Fubini-Pick tensor is

$$A_{ijk} = -\frac{1}{2} f_{ijk}.$$

Then the curvature tensor and the Ricci tensor are

$$\begin{aligned} R_{ijkl} &= \sum f^{mh} (A_{jkm} A_{hil} - A_{ikm} A_{hjl}) \\ R_{ik} &= \sum f^{mh} f^{jl} (A_{jkm} A_{hil} - A_{ikm} A_{hjl}). \end{aligned}$$

Let u be the Legendre transformation of f and $\Omega^* = \nabla^f(\Omega) \subset \mathfrak{t}^*$. Then it is known that $\nabla^f : (\Omega, G_f) \rightarrow (\Omega^*, G_u)$ is isometric.

Let $\rho = [\det(f_{ij})]^{-\frac{1}{n+2}}$, we introduce the following affine invariants:

$$\Phi = \frac{\|\nabla \rho\|_G^2}{\rho^2} \quad (2.1)$$

$$4n(n-1)J = \sum f^{il} f^{jm} f^{kn} f_{ijk} f_{lmn} = \sum u^{il} u^{jm} u^{kn} u_{ijk} u_{lmn}. \quad (2.2)$$

Φ is called the norm of the *Tchebychev vector field* and J is called the *Pick invariant*. Put

$$\Theta = J + \Phi. \quad (2.3)$$

2.2 Affine transformation rules

We study how the affine transformation affects the Calabi geometry.

Definition 2.1 *By an affine transformation, we mean a transformation as the following:*

$$\hat{A} : \mathfrak{t}^* \times \mathbb{R} \rightarrow \mathfrak{t}^* \times \mathbb{R}; \quad \hat{A}(\xi, \eta) = (A\xi, \lambda\eta),$$

where A is an affine transformation on \mathfrak{t}^* . If $\lambda = 1$ we call \hat{A} the *base-affine transformation*.

Let u be a function on \mathfrak{t}^* . Then \hat{A} induces an affine transformation on u :

$$u^*(\xi) = \lambda u(A^{-1}\xi).$$

We write u^* by $\hat{A}(u)$.

Then we have the following lemma of the *affine transformation rule* for the affine invariants.

Lemma 2.2 *Let $u^* = \hat{A}(u)$ be as above, then*

1. $\det(u_{ij}^*)(\xi) = \lambda^2 |A|^{-2} \det(u_{ij})(A^{-1}\xi)$.
2. $G_{u^*}(\xi) = \lambda G_u(A^{-1}\xi)$;
3. $\Theta_{u^*}(\xi) = \lambda^{-1} \Theta_u(A^{-1}\xi)$;
4. $S(u^*)(\xi) = \lambda^{-1} S(u)(A^{-1}\xi)$.

These can be easily verified. We skip the proof. As a corollary,

Lemma 2.3 *G and Θ are invariant with respect to the base-affine transformation. $\Theta \cdot G$ and $S \cdot G$ are invariant with respect to affine transformations.*

2.3 The differential inequality of Φ

We calculate $\Delta\Phi$ on the Riemannian manifold (Ω, G_f) and derive a differential inequality. We only need the inequality for the case $\mathcal{S}(f) = 0$ in this paper. This is already done in [34]. The following results can be find in [34] (See also [39, 47]).

Lemma 2.4 *The following two equations are equivalent*

$$\mathcal{S}(f) = 0, \quad (2.4)$$

$$\Delta\rho = \frac{n+4}{2} \frac{\|\nabla\rho\|_G^2}{\rho}. \quad (2.5)$$

Theorem 2.5 *Let f be a smooth strictly convex function on Ω with $\mathcal{S}(f) = 0$, then*

$$\begin{aligned} \Delta\Phi &\geq \frac{n}{2(n-1)} \frac{\|\nabla\Phi\|_G^2}{\Phi} + \frac{n^2-4}{n-1} \langle \nabla\Phi, \nabla \log \rho \rangle_G \\ &\quad + \frac{(n+2)^2}{2} \left(\frac{1}{n-1} - \frac{n-1}{4n} \right) \Phi^2. \end{aligned}$$

2.4 The equivalence between Calabi metrics and Euclidean metrics

In this subsection we prove the equivalence between Calabi metrics and Euclidean metrics under the condition that Θ is bounded above. The equivalence is based on Lemma 2.7 which gives an estimate for eigenvalues of (u_{ij}) . A similar estimate was first proved in [36], but the lemma given here is slightly stronger.

In this section, $D_a(p)$ and $B_a(p)$ denote the balls of radius a that is centered at p with respect to the Euclidean metric and Calabi metric respectively. Similarly, d_E and d_u are the distance functions with respect to the Euclidean metric and Calabi metric respectively.

First we prove the following lemma.

Lemma 2.6 *Let $\Gamma : \xi = \xi(t), t \in [0, t_0]$, be a curve from $\xi(0) = 0$ to $p_0 = \xi(t_0)$. Suppose that $\Theta \leq N^2$ along Γ . Then for any $p \in \Gamma$,*

$$\left| \frac{d \log T}{ds} \right| \leq 2n^2 \sqrt{N}, \quad \left| \frac{d \log \det(u_{ij})}{ds} \right| \leq (n+2) \sqrt{N}, \quad (2.6)$$

where $T = \sum u_{ii}$ and s denotes the arc-length parameter with respect to the metric G .

Proof. For any $p \in \Gamma$, by an orthogonal coordinate transformation, we can choose the coordinates ξ_1, \dots, ξ_n such that $u_{ij}(p) = \lambda_i \delta_{ij}$. Denote by λ_{max} (resp. λ_{min}) the maximal (resp. minimal) eigenvalue of (u_{ij}) . Suppose that, at p ,

$$\frac{\partial}{\partial t} = \sum a_i \frac{\partial}{\partial \xi_i}.$$

We have

$$\begin{aligned}
\frac{1}{T^2} \left(\frac{\partial T}{\partial t} \right)^2 &= \frac{1}{T^2} \left(\sum_{ij} u_{ij} a_j \right)^2 \leq \frac{n}{T^2} \sum_j \left(\sum_i u_{ij} \right)^2 a_j^2 \\
&\leq \frac{n}{T^2} \left(\sum_j \frac{1}{\lambda_j} \left(\sum_i u_{ij} \right)^2 \right) \left(\sum_j \lambda_j a_j^2 \right) \\
&\leq n^2 \left(\sum u^{im} u^{jn} u^{kl} u_{ijk} u_{mnl} \right) \left(\sum_{i,j} u_{ij} a_i a_j \right) \\
&\leq 4n^4 \Theta \sum_{i,j} u_{ij} a_i a_j = 4n^4 \Theta \cdot \left(\frac{ds}{dt} \right)^2.
\end{aligned}$$

Here we use the definition of Θ and J (cf. (2.2) and (2.3)). For the third " \leq ", we use the following computation:

$$\begin{aligned}
\sum_{i,j,k} u^{im} u^{jn} u^{kl} u_{ijk} u_{mnl} &= \sum_{i,j,k} u_{ijk} u_{mnl} \frac{1}{\lambda_l} \delta_l^k \frac{1}{\lambda_i} \delta_i^m \frac{1}{\lambda_j} \delta_j^n \\
&\geq \sum_{i,k} (u_{iik})^2 \frac{1}{\lambda_k \lambda_{max}^2} \geq \frac{1}{n} \frac{\sum_k \left(\sum_i u_{iik} \right)^2 \frac{1}{\lambda_k}}{\left(\sum_i \lambda_i \right)^2}.
\end{aligned}$$

Therefore, we have the differential inequality

$$\left| \frac{d \log T}{ds} \right| \leq 2n^2 \sqrt{\Theta} \leq 2n^2 N.$$

By the definition of Φ and (2.3) we have $\left| \frac{d \log \rho}{ds} \right| \leq N$. Then the lemma follows. q.e.d.

Lemma 2.7 *Let u be a smooth, strictly convex function defined in \mathbb{R}^n . Suppose that*

$$u_{ij}(0) = \delta_{ij}, \quad \Theta \leq N^2 \quad \text{in } \mathbb{R}^n. \quad (2.7)$$

Let $a \geq 1$ be a constant. Let $\lambda_{\min}(a), \lambda_{\max}(a)$ be the minimal and maximal eigenvalues of (u_{ij}) in $B_a(0)$. Then there exist constants $a_1, C_1 > 0$, depending only on N and n , such that

$$\text{(i)} \quad \exp(-C_1 a) \leq \lambda_{\min}(a) \leq \lambda_{\max}(a) \leq \exp(C_1 a), \quad \text{in } B_a(0),$$

$$\text{(ii)} \quad D_{a_1}(0) \subset B_a(0) \subset D_{\exp(C_1 a)}(0).$$

Proof. (ii) is the corollary of (i). We now prove (i). Let $\Gamma : \xi = \xi(t), t \in [0, t_0]$, be any geodesic from $\xi(0) = 0$ to $p_0 = \xi(t_0)$. Here t is the Euclid arc-length parameter of Γ from 0. By Lemma 2.6 we obtain

$$\left| \frac{d \log T}{ds} \right| \leq 2n^2 N, \quad \left| \frac{d \log \det(u_{ij})}{ds} \right| \leq (n+2)N.$$

where s denotes the arc-length parameter with respect to the metric G . By integrating with respect to s , we have

$$\begin{aligned} \frac{1}{n} \exp \{-2n^2 Na\} &\leq T(q) \leq n \exp \{2n^2 Na\} \\ \exp \{-(2n^2 + 2)Na\} &\leq \det(u_{ij})(q) \leq \exp \{(2n^2 + 2)Na\} \end{aligned}$$

for any $q \in B_a(0)$. The lemma then follows. ■

As a corollary, we have

Theorem 2.8 *Suppose that $\{u_k\}_{k=1}^\infty$ is a sequence of smooth strictly convex functions on $\Omega_k \subset \mathbb{R}^n$ containing 0 and $\Theta_{u_k} \leq N^2$; suppose that u_k are already normalized such that*

$$u_k \geq u_k(0) = 0, \quad \partial_{ij}^2(u_k)(0) = \delta_{ij}.$$

Then for any $a > 0$ with $B_{a,u_k}(0) \subset \Omega_k$, u_k locally C^2 -uniformly converges to a strictly convex function u_∞ in $B_{a,u_\infty}(0)$. Moreover, if (Ω_k, G_{u_k}) is complete, $(\Omega_\infty, G_{u_\infty})$ is complete.

Under the condition that Θ is bounded, the completeness of Calabi metrics and that of Euclidean metrics are equivalent.

Theorem 2.9 *Let $\Omega \subset \mathbb{R}^n$ be a convex domain. Let u be a smooth strictly convex function on Ω with $\Theta \leq N^2$. Then (Ω, G_u) is complete if and only if the graph of u is Euclidean complete in \mathbb{R}^{n+1} .*

Proof. " \implies ". Without loss of generality, we assume that $0 \in \Omega$ and

$$u \geq u(0), \quad u_{ij}(0) = \delta_{ij}.$$

We will prove that $u|_{\partial\Omega} = +\infty$. Namely, for any Euclidean unit vector v and the ray from 0 to $\partial\Omega$ along v direction, given by

$$\ell_v : [0, t) \rightarrow \Omega, \quad \ell_v(s) = sv, \quad \lim_{s \rightarrow t} \ell_v(s) \in \partial\Omega,$$

we have $u(\ell_v(s)) \rightarrow \infty$ as $s \rightarrow t$.

On $[0, t)$, we choose a sequence of points

$$0 = s_0 < s_1 < \cdots < s_k < \cdots$$

such that $d_u(\ell_v(s_i), \ell_v(s_{i+1})) = 1$. Since Ω is complete, the sequence is an infinite sequence. Set $u_i = u(\ell_v(s_i))$. We now claim that

Claim. There is a constant $\delta > 0$ such that $u_{i+1} - u_i \geq \delta$.

Proof of claim. We show this for $i = 0$. By Lemma 2.7, we conclude that $B_1(0)$ contains an Euclidean ball $D_{a_1}(0)$. Since the eigenvalue of (u_{ij}) are bounded from below and $\nabla u(0) = 0$, we conclude that there exists a constant δ

such that $u(\xi) - u(0) \geq \delta$ for any $\xi \in B_1(0) - D_{a_1(0)}$. In particular, $u_1 - u_0 \geq \delta$. Now consider any i . Let $p = \ell_v(s_i)$ and $q = \ell_v(s_{i+1})$. Since the claim is invariant with respect to the base-affine transformations, we first apply a base-affine transformation such that $u_{ij}(p) = \delta_{ij}$. Furthermore, we normalize u to a new function \tilde{u} such that p is the minimal point:

$$\tilde{u}(\xi) = u(\xi) - u(p) - \nabla u(p)(\xi - p).$$

Then by the same argument as $i = 0$ case, $\tilde{u}(q) - \tilde{u}(p) \geq \delta$. Moreover

$$u(q) - u(p) = \tilde{u}(q) - \tilde{u}(p) + \nabla u(p)(q - p) \geq \tilde{u}(q) - \tilde{u}(p) \geq \delta.$$

Here we use the convexity of u for the first " \geq ". End of the proof of the claim.

By the claim, $u_k \geq k\delta$ and goes to ∞ as $k \rightarrow \infty$.

" \Leftarrow ". We assume that $u \geq u(0) = 0$. Since the graph of u is Euclidean complete, for any $C > 0$, the section $S_u(0, C)$ is compact (the definition of section is given in §3.1). Consider the function

$$F = \exp \left\{ -\frac{2C}{C-u} \right\} \frac{\|\nabla u\|_G^2}{(1+u)^2}$$

defined in $S_u(0, C)$. F attains its maximum at some interior point p^* . We may assume $\|\nabla u\|_G(p^*) > 0$. Choose a frame field of the Calabi metric G around p^* such that

$$u_{ij}(p^*) = \delta_{ij}, \quad \|\nabla u\|_G(p^*) = u_1(p^*), \quad u_i(p^*) = 0, \quad i \geq 2.$$

Then at p^* , $F_i = 0$ implies that

$$\left(-\frac{2u_{,1}}{1+u} - \frac{2Cu_{,1}}{(C-u)^2} \right) u_{,1}^2 + 2u_{,1}u_{,11} = 0.$$

Using $u_1(p^*) > 0$ and $\frac{2C}{(C-u)^2} > 0$, we conclude that

$$\frac{u_{,1}^2}{(1+u)^2} \leq \frac{|u_{,11}|}{1+u}. \quad (2.8)$$

Note that

$$|u_{,11}| = |\Gamma_{11}^1 u_{,1} + u_{11}| = |1 + u_{111}u_{,1}| \leq 1 + |2n\sqrt{J}|u_{,1} \leq 1 + 2nNu_{,1}.$$

Substituting this into (2.8) and applying the Schwarz's inequality we have

$$\frac{u_{,1}^2}{(1+u)^2} \leq 2 + 8n^2N^2.$$

Then $F(p^*) \leq e^{-2}(2 + 8n^2N^2)$. Since F is maximum at p^* , we obtain that $F \leq C_1$. Hence in $S_u(0, \frac{C}{2})$,

$$\|\nabla \log(1+u)\|_G \leq C_2, \quad (2.9)$$

for some constant C_2 that is independent of C . Let $C \rightarrow \infty$, we have that (2.9) holds everywhere.

Take a point $p_1 \in \partial S_u(0, C)$ such that $d(0, p_1) = d(0, \partial S_u(0, C))$. Let l be the shortest geodesic from 0 to p_1 . Following from (2.9) we have

$$C_2 \geq \|\nabla \log(1 + u)\|_G \geq \left| \frac{d \log(1 + u)}{ds} \right|,$$

where s denotes the arc-length parameter with respect to the metric G . Applying this and a direct integration we obtain

$$d(0, p_1) = \int_l ds \geq C_2^{-1} \log(1 + u)|_{p_1} = C_2^{-1} \log(1 + C).$$

As $C \rightarrow \infty$, we obtain $d(0, \partial S_u(0, C)) \rightarrow +\infty$. Hence (Ω, G_u) is complete. q.e.d.

3 Convergence theorems and Bernstein Properties for Abreu equations

3.1 Basic Definitions

In this subsection we review basic concepts of convex domains and (strictly) convex functions.

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain. It is well-known (see [28], p.27) that there exists a unique ellipsoid E , which attains the minimum volume among all the ellipsoids that contain Ω and that is centered at the center of mass of Ω , such that

$$2^{-\frac{3}{2}}E \subset \Omega \subset E,$$

where $2^{-\frac{3}{2}}E$ means the $2^{-\frac{3}{2}}$ -dilation of E with respect to its center. Let T be an affine transformation such that $T(E) = D_1(0)$, the unit disk. Put $\tilde{\Omega} = T(\Omega)$. Then

$$2^{-\frac{3}{2}}D_1(0) \subset \tilde{\Omega} \subset D_1(0). \quad (3.1)$$

We call T the normalizing transformation of Ω .

Definition 3.1 *A convex domain Ω is called normalized when its center of mass is 0 and $2^{-\frac{3}{2}}D_1(0) \subset \Omega \subset D_1(0)$.*

Let $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be an affine transformation given by $A(\xi) = A_0(\xi) + a_0$, where A_0 is a linear transformation and $a_0 \in \mathbb{R}^n$. For any Euclidean vector a , denote by $|a|$ the Euclidean norm of the vector a . If there is a constant $L > 0$ such that $|a_0| \leq L$ and for any Euclidean unit vector v

$$L^{-1} \leq |A_0 v| \leq L,$$

we say that A is L -bounded.

Definition 3.2 A convex domain Ω is called L -normalized if its normalizing transformation is L -bounded.

The following lemma is useful to measure the normalization of a domain.

Lemma 3.3 Let $\Omega \subset \mathbb{R}^2$ be a convex domain. Suppose that there exists a pair of constants $R > r > 0$ such that

$$D_r(0) \subset \Omega \subset D_R(0),$$

then Ω is L -normalized, where L depends only on r and R .

Proof. Suppose the normalizing transformation $T(\xi) = A(\xi - p)$, where A is a linear transformation and $p \in \Omega$. Then

$$T(D_R(0)) \supset 2^{-\frac{3}{2}} D_1(0) \Rightarrow D_R(0) - p \supset A^{-1}(2^{-\frac{3}{2}} D_1(0)) \Rightarrow D_{2R}(0) \supset A^{-1}(2^{-\frac{3}{2}} D_1(0)).$$

Here, we use the fact $p \in D_R(0)$. Therefore for any Euclidean unit vector v

$$|Av| \geq 2^{-\frac{5}{2}} R^{-1}.$$

On the other hand,

$$T(D_r(0)) \subset D_1(0) \Rightarrow AD_r(0) - Ap \subset D_1(0)$$

In particular, we have $-Ap \in D_1(0)$. Then

$$AD_r(0) \subset D_2(0).$$

It follows that for any Euclidean unit vector v

$$|Av| \leq \frac{2}{r}.$$

We get the upper and lower bounds of eigenvalues of A . Furthermore $|Ap| \leq 1$. So T is L -bounded for some constant L . q.e.d.

Let Ω be a convex domain in \mathbb{R}^2 . We define its *uniform ξ_i -width* $wd_i(\Omega)$, $i = 1, 2$ by the following:

$$wd_1(\Omega) = \max_{t \in \mathbb{R}} \max_{\{\xi, \xi' \in \Omega | \xi_2 = \xi'_2 = t\}} |\xi_1 - \xi'_1|, \quad wd_2(\Omega) = \max_{t \in \mathbb{R}} \max_{\{\xi, \xi' \in \Omega | \xi_1 = \xi'_1 = t\}} |\xi_2 - \xi'_2|.$$

Then

Lemma 3.4 Let Ω be a bounded convex domain in \mathbb{R}^2 with $\Omega \subset D_b(0)$. Suppose that $wd_i(\Omega) \geq a > 0$. Then Ω is L -normalized, where L depends only on a and b .

Proof. Since wd_i has lower bound, there exists a disk of radius r in Ω , where r depends only on a . Let p_0 be the center of disk. Then by the upper bound of wd_i it is obvious that $\Omega \subset D_{p_0}(2b)$. Then this lemma is a corollary of Lemma 3.3. q.e.d.

Let u be a convex function on Ω . Let $p \in \Omega$. Consider the set

$$\{\xi \in \Omega | u(\xi) \leq u(p) + \nabla u(p) \cdot (\xi - p) + \sigma\}.$$

If it is compact in Ω , we call it *a section of u at p with height σ* and denote it by $S_u(p, \sigma)$.

In this section, we need an important result from the classical convex body theory (see [5]). Let M be a convex hypersurface in \mathbb{R}^{n+1} and e be a subset of M . We denote by $\psi_M(e)$ the spherical image of e . Denote by $\sigma_M(e)$ the area (measure) of the spherical image $\psi_M(e)$, denote by $A(e)$ the area of the set e on M . The ratio $\sigma_M(e)/A(e)$ is called the specific curvature of e . In the case $n = 2$, the following theorem holds (see [5]):

Theorem 3.5 (Alexandrov-Pogorelov) : *A convex surface whose specific curvature is bounded away from zero is strictly convex.*

3.2 The convergence theorem for Abreu equations

In this subsection we prove a very useful convergence theorem for Abreu equations(cf. Theorem 3.6). Essentially, there are two types of convergence theorems in this paper for the Abreu equations. One is to get the convergence by controlling Θ (cf. Theorem 2.8), the other is to get the convergence by controlling the sections. Here, we explain the later one.

Denote by $\mathcal{F}(\Omega, C)$ the class of convex functions defined on Ω such that

$$\inf_{\Omega} u = 0, \quad u = C > 0 \text{ on } \partial\Omega,$$

and

$$\mathcal{F}(\Omega, C; K_o) = \{u \in \mathcal{F}(\Omega, C) | |\mathcal{S}(u)| \leq K_o\}.$$

We will assume that $\Omega \subset \mathbb{R}^2$ is normalized in this subsection.

The main result of this subsection is the following convergence theorem.

Theorem 3.6 *Let $\Omega \subset \mathbb{R}^2$ be a normalized domain. Let $u_k \in \mathcal{F}(\Omega, 1; K_o)$ be a sequence of functions and p_k^o be the minimal point of u_k . Then there exists a subsequence of functions, without loss of generality, still denoted by u_k , converges to a function u_∞ and p_k^o converges to p_∞^o satisfying:*

(i) *there are constants s and C_2 such that $d_E(p_k^o, \partial\Omega) > 2s$, and in $D_s(p_\infty^o)$*

$$\|u_k\|_{C^{3,\alpha}} \leq C_2$$

for any $\alpha \in (0, 1)$; in particular, u_k $C^{3,\alpha}$ -converges to u_∞ in $D_s(p_\infty^o)$.

(ii) *there is a constant $\delta \in (0, 1)$, such that $S_{u_k}(p_k^o, \delta) \subset D_s(p_\infty^o)$.*

(iii) there exists a constant $\mathbf{b} > 0$ such that $S_{u_k}(p_k^o, \delta) \subset B_{\mathbf{b}}(p_k^o)$.

Furthermore, if u_k is smooth and C^∞ -norms of $\mathcal{S}(u_k)$ are uniformly bounded, then u_k smoothly converges to u_∞ in $D_{\mathbf{s}}(p_\infty^o)$.

Remark 3.7 Let Ω_k be a sequence of L -normalized convex domain and $u_k \in \mathcal{F}(\Omega_k, C_k; K_o)$ be a sequence of functions. If the sequence of constants C_k satisfies $C^{-1} \leq C_k \leq C$ for some constant $C > 0$, then Theorem 3.6 still holds.

Remark 3.8 This theorem with $K_o = 0$ has been proved in [39], the proof is pure analysis and the calculation is very long. The authors of [39] have pointed out that one can use the Alexandrov-Pogorelov Theorem (Theorem 3.5) to give a simpler proof. Here we use this idea to give the proof for $K_o \neq 0$.

We prove the following proposition, which is equivalent to Theorem 3.6.

Proposition 3.9 Let $\Omega \subset \mathbb{R}^2$ be a normalized domain. Let $u \in \mathcal{F}(\Omega, 1; K_o)$ and p^o be its minimal point. Then

(i) there are constants \mathbf{s} and \mathbf{C}_2 such that $d_E(p^o, \partial\Omega) > \mathbf{s}$ and in $D_{\mathbf{s}}(p^o)$

$$\|u\|_{C^{3,\alpha}} \leq \mathbf{C}_2$$

for any $\alpha \in (0, 1)$;

(ii) there is a constant $\delta \in (0, 1)$, such that $S_u(p^o, \delta) \subset D_{\mathbf{s}}(p^o)$.

(iii) there exists a constant $\mathbf{b} > 0$ such that $S_u(p^o, \delta) \subset B_{\mathbf{b}}(p^o)$.

In the statement, all the constants only depend on K_o .

Furthermore, if u is smooth, then

$$\|u\|_{C^\infty(D_\epsilon(p^o))} \leq \mathbf{C}_3$$

where \mathbf{C}_3 depends on the C^∞ -norm of $\mathcal{S}(u)$.

To prove Proposition 3.9, we need some useful lemmas for functions in $\mathcal{F}(\Omega, 1; K_o)$.

Proposition 3.10 Let $u \in \mathcal{F}(\Omega, 1; K_o)$. If there is a constant $C_1 > 0$ such that in Ω

$$C_1^{-1} \leq \det(u_{ij}) \leq C_1. \quad (3.2)$$

Then for any $\Omega^* \subset \subset \Omega$, $p > 2$, we have the estimate

$$\|u\|_{W^{4,p}(\Omega^*)} \leq C, \quad \|u\|_{C^{3,\alpha}(\Omega^*)} \leq C, \quad (3.3)$$

where C depends on $n, p, C_1, K_o, \text{dist}(\Omega^*, \partial\Omega)$.

Proof. In [8] Caffarelli-Gutierrez proved a Hölder estimate of $\det(u_{ij})$ for homogeneous linearized Monge-Ampere equation assuming that the Monge-Ampère measure $\mu[u]$ satisfies some condition, which is guaranteed by (3.2). When $f \in L^\infty$, following their argument one can obtain Hölder continuity of $\det(u_{ij})$. By Caffarelli's $C^{2,\alpha}$ estimates for Monge-Ampere equation ([7]) we have

$$\|u\|_{C^{2,\alpha}(\Omega^*)} \leq C_2.$$

Following from the standard elliptic regularity theory we have $\|u\|_{W^{4,p}(\Omega^*)} \leq C$. By Sobolev embedding theorem we have

$$\|u\|_{C^{3,\alpha}(\Omega^*)} \leq C_2 \|u\|_{W^{4,p}(\Omega^*)}.$$

Then we can obtain the proposition. q.e.d.

Lemma 3.11 *Let $u \in \mathcal{F}(\Omega, 1; K_o)$. Let $\Omega^* = \nabla^u(\Omega)$ and $f = L(u)$. For any $\Omega_1^* \subset \subset \Omega^*$ there is a constant $d > 0$ such that $\det D^2 u > d$ on $\Omega_1 = \nabla^f(\Omega_1^*)$, where d depends on $\Omega, d_E(\Omega_1^*, \partial\Omega^*), \text{diam}(\Omega_1^*)$ and K_o .*

This is proved by Li-Jia in [39].

Lemma 3.12 *Let $u \in \mathcal{F}(\Omega, 1; K_o)$. There exists two constants $d_1, d_2 > 0$ such that*

$$\exp \left\{ -\frac{4}{1-u} \right\} \frac{\det(u_{ij})}{(d_1 + f)^4} \leq d_2. \quad (3.4)$$

This is proved by Chen-Li-Sheng in [10].

For $u \in \mathcal{F}(\Omega, 1; K_o)$ with $u(p_o) = 0$. Choose R such that

$$D_R(p_o) \supset \Omega \supset S_u(p_o, \frac{1}{2}). \quad (3.5)$$

Then we have the following lemma.

Lemma 3.13 $\nabla^u(S_u(p_o, \frac{1}{2}))$ contains the disk $D_r(0), r = (2R)^{-1}$. Moreover, in $D_r(0)$ (i) $-1 \leq f \leq r$; (ii) $\det(f_{ij}) \geq d'$ for some constant d' depending only on K_o .

Proof. The first statement follows from (3.5). At $\nabla^u(0)$,

$$f_1(\nabla^u(0)) = f_2(\nabla^u(0)) = 0, \quad u(0) + f(\nabla^u(0)) = 0.$$

By the convexity of f , we have $\inf f = f(\nabla^u(0))$. Then

$$f \geq f(\nabla^u(0)) = -u(0) \geq -1.$$

Since $f(x) + u(\xi) = x \cdot \xi, u \geq 0$ and $\Omega \subset D_1(0)$, we have

$$f(x) \leq x \cdot \xi \leq r \cdot 1 = r.$$

From (3.4) and (i), we know that $\det(u_{ij})$ is bounded from above in D_r . Therefore $\det(f_{ij}) = [\det(u_{ij})]^{-1}$ is bounded from below. q.e.d

Lemma 3.14 *Use the notations in Lemma 3.13. Then for any $\alpha \in (0, 1)$, there is a constant C_2 such that in $D_{\frac{r}{2}}(0)$*

$$\|f\|_{C^{3,\alpha}} \leq C_2, \quad (3.6)$$

and for any eigenvalue ν_f of (f_{ij})

$$C_2^{-1} \leq \nu_f \leq C_2. \quad (3.7)$$

Proof. If not, we have a sequence of u_k such that in $D_{\frac{r}{2}}(0)$

$$\|f_k\|_{C^{3,\alpha}} \rightarrow \infty.$$

By (i) of Lemma 3.13, we know that f_k locally uniformly converges to a convex function f_∞ on $D_r(0)$; and by (ii) of Lemma 3.13 and Theorem 3.5, we conclude that f_∞ is *strictly convex* on $D_{\frac{r}{2}}(0)$. Hence there is a constant b_0 such that $S_{f_\infty}(0, b_0)$ is compact in $D_{\frac{r}{2}}(0)$. Moreover, in this section, $\det(D^2 f_\infty)$ is bounded from above (by Lemma 3.11) and below (by (ii) of Lemma 3.13). Hence by Proposition 3.10, we conclude that the $C^{3,\alpha}$ -norms of f_k in the section $S_{f_k}(0, \frac{1}{2}b_0)$ are uniformly bounded by a constant N that is independent of k . Instead of considering 0, we repeat the argument for any $x \in D_{\frac{r}{2}}(0)$ we conclude that the $C^{3,\alpha}$ -norms of f_k are uniformly bounded in some section $S_{f_k}(x, b_x)$ for some b_x . By the compactness of $D_{\frac{r}{2}}(0)$, we conclude that the $C^{3,\alpha}$ -norms of f_k in $D_{\frac{r}{2}}(0)$ are uniformly bounded. This contradicts to the assumption. q.e.d.

Proof of Proposition 3.9. From the lower bound of eigenvalues in Lemma 3.14, there is a constant b such that for any f

$$S_f(0, b) \subset \subset D_{\frac{r}{2}}(0).$$

Let $V_f = \nabla^f(S_f(0, b))$. Since f is $C^{2,\alpha}$ -bounded. V_f must contain a disk $D_\epsilon(p_o)$, where ϵ is independent of f . The regularity of u in V_f follows from Proposition 3.10. (iii) is a direct consequence of (i) and (ii). q.e.d.

3.3 Bernstein properties for Abreu equations

When $\mathcal{S}(u) = 0$, we expect that u must be quadratic polynomial under certain completeness assumptions. This is called the Bernstein property.

We state the theorems here. The proof of the theorem can be found in [34].

Theorem 3.15 *[Jia-Li] Let $\Omega \subset \mathbb{R}^n$ and $f : \Omega \rightarrow \mathbb{R}$ be a smooth strictly convex function. Suppose that (Ω, G_f) is complete and*

$$\mathcal{S}(f) = - \sum f^{ij} (\log \det(D^2 f))_{ij} = 0$$

If $n \leq 5$, then the graph of f must be an elliptic paraboloid.

Remark 3.16 *If we only assume that $f \in C^5$, Theorem 3.15 remains hold.*

Now we use the convergence theorem 3.6 to prove the following theorem.

Theorem 3.17 [Li-Jia] *Let $u(\xi_1, \xi_2)$ be a C^∞ strictly convex function defined in a convex domain $\Omega \subset \mathbb{R}^2$. If*

$$\mathcal{S}(u) = 0, \quad u|_{\partial\Omega} = +\infty,$$

then the graph of u must be an elliptic paraboloid.

Proof: We show $\Phi \equiv 0$. If the theorem is not true, then there is a point $p \in M$ such that $\Phi(p) > 0$. By subtracting a linear function we may suppose that $u(\xi) \geq u(p) = 0$.

Choose a sequence $\{C_k\}$ of positive numbers such that $C_k \rightarrow \infty$ as $k \rightarrow \infty$. For any $C_k > 0$ the section $S_u(p, C_k)$ is a bounded convex domain. Let

$$u_k(\xi) = C_k^{-1}u(\xi), \quad k = 1, 2, \dots$$

Then

$$\Phi_{u_k}(p) = C_k \Phi(p) \rightarrow \infty, \quad k \rightarrow \infty. \quad (3.8)$$

Let T_k be the normalizing transformation of $S_u(p, C_k) = S_{u_k}(p, 1)$. We obtain a sequence of convex functions

$$\tilde{u}_k(\xi) := u_k(T_k^{-1}(\xi)).$$

Let $\tilde{p}_k = T_k(p)$. By Theorem 3.6 we conclude that $\tilde{u}^{(k)}$ converges with all derivatives to a smooth and strictly convex function u_∞ in a neighborhood of \tilde{p}_∞ , which is the limit of \tilde{p}_k . Therefore $\Phi_{u_k}(p) = \Phi_{\tilde{u}_k}(\tilde{p}_k)$ (cf. Lemma 2.2) is uniformly bounded, which contradicts to (3.8). ■

4 Affine blow-up analysis and interior estimates of Θ

When considering a sequence of functions with poor convergence behavior, we may normalize the domain and the functions such that the new sequence converges nicely. Such an argument is called the blow-up analysis. In this paper, we apply the affine transformations (cf. Definition 2.1) for normalization. This is a very powerful tool to estimate *affine invariants* in various circumstances. Hence, we call the technique to be the affine blow-up analysis.

4.1 Affine Blow-up analysis

Let \mathcal{M} be a set of smooth strictly convex functions on Ω . For simplicity, we suppose that (Ω, G_u) is complete for any $u \in \mathcal{M}$.

Given $u \in \mathcal{M}$, let $Q_u : \Omega \rightarrow \mathbb{R}$ be a non-negative function which is invariant under base-affine transformations (cf Definition 2.1). Suppose we want to prove that there is a constant N , s.t. for any $u \in \mathcal{M}$

$$Q_u \leq N < \infty. \quad (4.1)$$

We usually argue by assuming that it is not true. Then there will be a sequence of functions u_k and a sequence of points u_k such that

$$Q_{u_k}(p_k) \rightarrow \infty. \quad (4.2)$$

Usually, we perform affine transformation to u_k and get a sequence of normalized function \tilde{u}_k to get contradictions (this is already used in the proof of Theorem 3.17). However, sometimes, we need a more tricky argument to get \tilde{u}_k . This is explained in the following remark.

Remark 4.1 *We explain a standard blow-up argument used in our papers. Suppose that (4.2) happens.*

- (i) *consider the function $F_k := Q_{u_k}(p)d_{u_k}^\alpha(p, \partial B_{p_k}(1))$ in the unit geodesic ball $B_1(p_k)$.*

Suppose that Qd^α is invariant with respect to affine transformations in Definition 2.1.

- (ii) *suppose that F_k attains its maximal at p_k^* . Set*

$$d_k = \frac{1}{2}d(p_k^*, \partial B_1(p_k)).$$

Then we conclude that

- $Q_{u_k}(p_k^*)d_k^\alpha \rightarrow \infty$.
- $Q_{u_k} \leq 2^\alpha Q_{u_k}(p_k^*)$ in $B_{d_k}(p_k^*)$.

We now restrict ourself on $B_{d_k}(p_k^)$;*

- (iii) *without loss of generality, by translation, we set p_k^* to 0. Now we re-normalize functions u_k by proper affine transformations such that for new functions, denoted by \tilde{u}_k , satisfy*

$$Q_{\tilde{u}_k}(0) = 1.$$

Then the ball $B_{d_k}(0)$ is normalized to a ball $B_{\tilde{d}_k}(0)$, where $\tilde{d}_k \rightarrow \infty$.

To summarize, we have a sequence of normalized functions \tilde{u}_k and we will study the convergence of the sequence to get contradiction. Usually, we conclude that \tilde{u}_k uniformly converges to \tilde{u}_∞ in a neighborhood of 0 and $Q_{\tilde{u}_\infty} \equiv 0$ which contradicts to the fact

$$\lim_{k \rightarrow \infty} Q_{\tilde{u}_k}(0) = 1.$$

In this final step, we need the aid of convergence theorems.

4.2 Interior estimate of Θ

The purpose of this section is to estimate Θ in a geodesic ball of the boundary $\partial\Delta$. We use the blow-up analysis explained in Remark 4.1 to prove the following result.

Proposition 4.2 *Let u be a smooth strictly convex function on a bounded convex domain $\Omega \subset \mathbb{R}^2$ with $\|\mathcal{S}(u)\|_{C^2(\Omega)} < K_o$, where $\|\cdot\|_{C^2}$ denotes the Euclidean C^2 -norm. Suppose that for any $p \in \Omega$,*

$$d_u(p, \partial\Omega) < N \quad (4.3)$$

for some constant $N > 0$. Then there is a constant $C_5 > 0$, depending only on Ω, N and K_o , such that

$$\Theta(p)d_u^2(p, \partial\Omega) \leq C_5, \quad \forall p \in \Omega. \quad (4.4)$$

Here $d_u(p, \partial\Omega)$ is the distance from p to $\partial\Omega$ with respect to the Calabi metric G_u .

Proof: If not, then there exist a sequence of functions u_k and a sequence of points p_k such that

$$\Theta_{u_k}(p_k)d_{u_k}^2(p_k, \partial\Omega) \rightarrow \infty.$$

Let $B^{(k)}$ be the $\frac{1}{2}d_{u_k}(p_k, \partial\Omega)$ -ball centered at p_k and consider the *affine-transformation-invariant function*(cf. Lemma 2.3)

$$F_k(p) = \Theta_{u_k}(p)d_{u_k}^2(p, \partial B^{(k)}).$$

F_k attains its maximum at p_k^* . Put

$$d_k = \frac{1}{2}d_{u_k}(p_k^*, \partial B^{(k)}).$$

By adding linear functions we assume that

$$u_k(p_k^*) = 0, \quad \nabla u_k(p_k^*) = 0.$$

By taking a coordinate translation we may assume that p_k^* has the coordinate 0. Then (cf. (ii) in Remark 4.1)

- $\Theta_{u_k}(0)d_k^2 \rightarrow \infty$.
- $\Theta_{u_k} \leq 4\Theta_{u_k}(0)$ in $B_{d_k}^{(k)}(0)$.

By (4.3) we have

$$\lim_{k \rightarrow \infty} \Theta_{u_k}(0) = +\infty. \quad (4.5)$$

We take an affine transformation on u_k :

$$\xi^* = A_k \xi, \quad u_k^*(\xi^*) := \lambda_k u_k(A_k^{-1} \xi^*),$$

where $\lambda_k = \Theta_{u_k}(0)$. Choose A_k such that $\partial_{ij}^2 u_k^*(0) = \delta_{ij}$. Denote $A_k^{-1} = (b_{ij}^k)$. Then by the affine transformation rule $\Theta_{u_k^*}(0) = 1$, and for any fixed large R , when k large enough

$$\Theta_{u_k^*} \leq 4 \quad \text{in} \quad B_R^{(k)}(0).$$

Moreover,

$$\mathcal{S}(u_k^*) = \frac{\mathcal{S}(u_k)}{\lambda_k} \rightarrow 0.$$

By Theorem 2.8 and Proposition 3.10, one concludes that u_k^* locally uniformly C^3 -converges to a function u_∞^* . In particular, there is a Euclidean ball $D_\epsilon(0)$ such that $D_\epsilon(0) \subset A_k(\Omega)$ as k large enough, thus $A_k^{-1}D_\epsilon(0) \subset \Omega$. It follows that for any $1 \leq i, j \leq 2$

$$|b_{ij}^k| \leq 2 \text{diam}(\Omega) \epsilon^{-1}, \quad (4.6)$$

as k large enough. By a direct calculation we have

$$\left| \frac{\partial \mathcal{S}(u_k^*)}{\partial \xi_i^*} \right| = \lambda_k^{-1} \left| \sum_j b_{ij}^k \frac{\partial \mathcal{S}(u_k)}{\partial \xi_j} \right| \leq 8 \text{diam}(\Omega) \epsilon^{-1} \lambda_k^{-1} K_o.$$

Similarly, we have

$$\|\mathcal{S}(u_k^*)\|_{C^2} \leq 128[(\text{diam}(\Omega))^2 \epsilon^{-2} + 1] \lambda_k^{-1} K_o.$$

Then by the bootstrap technique we obtain that u_k^* locally uniformly C^5 -converges to u_∞^* with

$$\mathcal{S}(u_\infty^*) = 0, \quad \nabla \mathcal{S}(u_\infty^*) = 0, \quad \Theta_{u_\infty^*}(0) = 1.$$

Hence by Theorem 3.15 and Remark 3.16, u_∞^* must be quadratic and $\Theta \equiv 0$. We get a contradiction. q.e.d.

Theorem 4.3 *Let u be a smooth strictly convex function on a bounded convex domain $\Omega \subset \mathbb{R}^2$ with $\|\mathcal{S}(u)\|_{C^2(\Omega)} < K_o$, where $\|\cdot\|_{C^2}$ denotes the Euclidean C^2 -norm. Assume that for any $p \in \Omega$,*

$$d_u(p, \partial\Omega) < \infty. \quad (4.7)$$

Suppose that the height of u is bounded, i.e.,

$$\max_{\Omega} u - \min_{\Omega} u \leq C_4 \quad (4.8)$$

for some constant $C_4 > 0$. Then there is a constant $C_5 > 0$, depending only on Ω, C_4 and K_o , such that

$$\Theta(p) d_u^2(p, \partial\Omega) \leq C_5, \quad \forall p \in \Omega. \quad (4.9)$$

Here $d_u(p, \partial\Omega)$ is the distance from p to $\partial\Omega$ with respect to the Calabi metric G_u .

Proof. We use affine blow-up analysis as in Theorem 4.2. The arguments before (4.5) is the same. Then by (4.8) we have

$$\inf_{\partial\Omega} u_k - u_k(p_k^*) \leq C_4. \quad (4.10)$$

If (4.5) holds, then we obtain corollary as in Proposition 4.2. Now we assume that $\Theta_{u_k}(0)$ is bounded above by some constant C_1 . Then $d_k \rightarrow \infty$. By a base-affine transformation we can assume that $\partial_{ij}^2 u_k(0) = \delta_{ij}$. Hence we conclude that u_k locally uniformly C^2 -converges to a function u_∞ and its domain Ω^∞ is complete with respect to G_{u_∞} . Then by Theorem 2.9, the graph of u_∞ is Euclidean complete. This contradicts to the equation (4.10). q.e.d.

Remark 4.4 *The condition (4.7) in Theorem 4.3 is not necessary. In fact, if the condition (4.7) does not hold, then (Ω, G_u) is complete; by using the blow-up analysis we can prove that in Ω ,*

$$\Theta \leq C$$

for some constant $C > 0$; by $\Theta \leq C$ and Theorem 2.9 we conclude that the graph of u is Euclidean complete in \mathbb{R}^3 . It contradicts to (4.8).

Similarly, we can prove that

Corollary 4.5 *Let u be as that in Theorem 4.3, with one modification:*

$$\|\mathcal{S}(u)\|_{C^3(\Omega)} < K_o.$$

Then there is a constant $C_5 > 0$, depending only on Ω, C_4 and K_o , such that

$$\mathcal{K}(p)d_u^2(p, \partial\Omega) \leq C_5, \quad \forall p \in \Omega. \quad (4.11)$$

Proof. We replace Θ by $\Theta + \mathcal{K}$ for the proof of Proposition 4.2 and Theorem 4.3. Follow word by word until the very last step. We need $|\mathcal{S}(u)|_{C^3}$ to conclude the converges of \mathcal{K} , then we get $(\Theta + \mathcal{K})_{u_\infty^*}(0) \cong 1$ by convergence. This contradicts to the Bernstein property as before. q.e.d.

In the proof, we use the fact that Θ and \mathcal{K} share the same affine transformation rule. Though $\|\nabla \log \mathcal{S}(u)\|^2$ share the same affine transformation rule as them, the proof can not go through since it is not well defined when $\mathcal{S}(u) = 0$. We need more arguments.

Corollary 4.6 *Let u be as that in Theorem 4.3, with one extra condition:*

$$|\mathcal{S}(u)| \geq \delta > 0.$$

Then there is a constant $C_5 > 0$, depending only on Ω, C_4 and K_o , such that

$$\|\nabla \log \mathcal{S}(u)\|^2(p)d_u^2(p, \partial\Omega) \leq C_5, \quad \forall p \in \Omega. \quad (4.12)$$

Proof. Let $F(p) := \|\nabla \log \mathcal{S}(u)\|^2(p) d^2(p, \partial\Omega)$. If F has no uniform upper bound, then there is a sequence u_k of functions such that $F_k(p_k) = \max_{\Omega} F_k$ and $\lim_{k \rightarrow \infty} F_k(p_k) = \infty$. Since F_k is an affine invariant function, by a sequence of affine transformations

$$\xi^* = A_k \xi, \quad u_k^*(\xi^*) := \lambda_k u_k(A_k^{-1} \xi^*),$$

we can assume that

$$d(p_k^*, \partial\Omega) = 1, \quad \partial_{ij}^2 u_k^*(p_k^*) = \delta_{ij}, \quad u^* \geq u^*(p_k^*) = 0.$$

Using Theorem 4.3, we have $\Theta(u_k^*) \leq 4C_5$ in $B_{\frac{1}{2}}(p_k^*)$. By the same argument of Proposition 4.2, we conclude that p_k^* converges to p_∞ and u_k $C^{3,\alpha}$ -converges to u in $D_\epsilon(p_\infty)$, and (4.6) still holds. Since

$$\|\nabla \log \mathcal{S}(u^*)\|_{u^*}^2 = \|\nabla \log \mathcal{S}(u)\|_{u^*}^2 = \sum u^{*ij} \frac{\partial \xi_k}{\partial \xi_i^*} \frac{\partial \xi_l}{\partial \xi_j^*} \frac{\partial \log \mathcal{S}(u)}{\partial \xi_k} \frac{\partial \log \mathcal{S}(u)}{\partial \xi_l},$$

by (4.6), $\|\mathcal{S}(u)\|_{C^2(\Omega)} \leq K_o$ and $|\mathcal{S}(u)| \geq \delta$ we conclude that F_k has uniform upper bound. It contradicts to the assumption. q.e.d.

5 Complex differential inequalities

In this section we extend the affine techniques to real functions defined on a domain $\Omega \subset \mathbb{C}^n$. Denote

$$\mathcal{R}^\infty(\Omega) := \{f \in C^\infty(\Omega) \mid f \text{ is a real function and } (f_{i\bar{j}}) > 0\},$$

where $(f_{i\bar{j}}) = \left(\frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}\right)$. For $f \in \mathcal{R}^\infty(\Omega)$, (Ω, ω_f) is a Kähler manifold.

For the sake of notations, we set

$$W = \det(f_{s\bar{t}}), \quad V = \log \det(f_{s\bar{t}}), \quad (5.1)$$

Introduce the functions

$$\Psi = \|\nabla V\|_f^2, \quad P = \exp(\kappa W^\alpha) \sqrt{W} \Psi, \quad (5.2)$$

Note that Ψ is the complex version of Φ in Calabi geometry. Constants κ and α in P are to be determined in (6.4). Denote

$$\|V_{,i\bar{j}}\|_f^2 = \sum f^{i\bar{j}} f^{k\bar{l}} V_{i\bar{l}} V_{k\bar{j}}, \quad \|V_{,ij}\|_f^2 = \sum f^{i\bar{j}} f^{k\bar{l}} V_{,ik} V_{,l\bar{j}}.$$

Denote by $\square = \sum f^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j}$ the Laplacian operator. Recall that

$$\mathcal{S}(f) = -\square V = -\sum f^{i\bar{j}} V_{i\bar{j}} \quad (5.3)$$

is the scalar curvature of ω_f .

5.1 Differential inequality-I

We are going to calculate $\square P$ and derive a differential inequality. This inequality is similar to that in Theorem 2.5.

Lemma 5.1 (Inequality-I)

$$\frac{\square P}{P} \geq \frac{\|V_{,i\bar{j}}\|_f^2}{2\Psi} + \alpha^2 \kappa (1 - 2\kappa W^\alpha) W^\alpha \Psi - \frac{2|\langle \nabla \mathcal{S}, \nabla V \rangle|}{\Psi} - \left(\alpha \kappa W^\alpha + \frac{1}{2}\right) \mathcal{S}, \quad (5.4)$$

where \langle, \rangle denotes the inner product with respect to the metric ω_f .

Proof. By definition,

$$\Psi_{,k} = \sum f^{i\bar{j}} (V_{,i} V_{,\bar{j}k} + V_{,ik} V_{,\bar{j}}),$$

$$\square \Psi = \sum f^{i\bar{j}} f^{k\bar{l}} (V_{,i} V_{,\bar{j}k\bar{l}} + V_{,ik\bar{l}} V_{,\bar{j}} + V_{,ik} V_{,\bar{j}\bar{l}} + V_{,i\bar{l}} V_{,\bar{j}k}).$$

By the Ricci identities

$$V_{,\bar{j}k\bar{l}} = V_{,k\bar{l}\bar{j}}, \quad V_{,ik\bar{l}} = V_{,k\bar{l}i} + \sum f^{m\bar{h}} V_{,m} R_{k\bar{h}i\bar{l}},$$

we have

$$\square \Psi = \sum f^{i\bar{j}} f^{k\bar{l}} (V_{,ik} V_{,\bar{j}\bar{l}} + V_{,i\bar{l}} V_{,\bar{j}k} - V_{,i\bar{l}} V_{,k} V_{,\bar{j}}) - 2\text{Re}(\sum f^{i\bar{j}} V_{,i} \mathcal{S}_{,\bar{j}}), \quad (5.5)$$

where we use the facts $R_{i\bar{j}} = -V_{,\bar{i}j}$ and $\square V = -\mathcal{S}$. Denote $\Pi = (\alpha \kappa W^\alpha + \frac{1}{2})$. Then

$$\begin{aligned} P_{,i} &= P \left(\frac{\Psi_{,i}}{\Psi} + \Pi V_{,i} \right) =: P \Lambda_i, \\ \square P &= P \left[\sum f^{i\bar{j}} \Lambda_i \Lambda_{\bar{j}} + \frac{\square \Psi}{\Psi} - \frac{\|\nabla \Psi\|_f^2}{\Psi^2} + \Pi \square V + \alpha^2 \kappa W^\alpha \Psi \right]. \end{aligned} \quad (5.6)$$

Choose a new complex coordinate system such that, at p ,

$$f_{i\bar{j}} = c \delta_{ij}, \quad V_1 = V_{\bar{1}}, \quad V_i = V_{\bar{i}} = 0 \quad \forall i > 1,$$

where $c = [W(p)]^{\frac{1}{n}}$. Then (5.6) can be re-written as

$$\begin{aligned} \square P &= \frac{P}{\Psi} [\square \Psi + \alpha^2 \kappa W^\alpha c^{-2} V_1^2 V_{\bar{1}}^2 + \Pi^2 c^{-2} V_1^2 V_{\bar{1}}^2 - \Pi \mathcal{S} \Psi \\ &\quad + 2\Pi c^{-2} (\text{Re}(V_{,1\bar{1}} V_{,\bar{1}}^2) + V_{,1\bar{1}} V_1 V_{\bar{1}})], \end{aligned} \quad (5.7)$$

here and later we denote Re the real part. From (5.5) we have

$$\begin{aligned} \square \Psi &= \sum c^{-2} (V_{ik} V_{\bar{i}\bar{k}} + V_{i\bar{k}} V_{\bar{i}k}) - c^{-2} V_{1\bar{1}} V_1 V_{\bar{1}} - 2\text{Re}(\sum f^{i\bar{j}} V_{,i} \mathcal{S}_{,\bar{j}}) \\ &\geq c^{-2} V_{1\bar{1}} V_{1\bar{1}} + c^{-2} V_{1\bar{1}}^2 + \sum_{i>1} c^{-2} V_{i\bar{k}} V_{\bar{i}k} - c^{-2} V_{1\bar{1}} V_1 V_{\bar{1}} \\ &\quad - 2\text{Re}(\sum f^{i\bar{j}} V_{,i} \mathcal{S}_{,\bar{j}}). \end{aligned} \quad (5.8)$$

Substituting (5.8) into (5.7), we have

$$\begin{aligned}
\Box P &\geq \frac{P}{c^2\Psi} [\Pi^2(V_1 V_{\bar{1}})^2 + 2\Pi Re(V_{,11} V_{\bar{1}}^2) + V_{11} V_{\bar{1}\bar{1}}] \\
&\quad - \frac{P}{\Psi} \left[2 \left| \sum f^{i\bar{j}} V_{,i} \mathcal{S}_{,\bar{j}} \right| + \Pi \mathcal{S} \Psi \right] + \frac{P}{c^2\Psi} \sum_{i>1} V_{i\bar{j}} V_{j\bar{i}} \\
&\quad + \frac{P}{c^2\Psi} [V_{1\bar{1}}^2 + \alpha^2 \kappa W^\alpha (V_1 V_{\bar{1}})^2 + 2(\alpha \kappa W^\alpha) V_{1\bar{1}} V_1 V_{\bar{1}}] \\
&= \frac{P}{c^2\Psi} \left[I_1 + I_2 + \sum_{i>1} V_{i\bar{j}} V_{j\bar{i}} \right] - \frac{P}{\Psi} \left[\Pi \mathcal{S} \Psi + 2 \left| \sum f^{i\bar{j}} V_{,i} \mathcal{S}_{,\bar{j}} \right| \right],
\end{aligned}$$

where

$$\begin{aligned}
I_1 &:= [\Pi^2(V_1 V_{\bar{1}})^2 + 2\Pi Re(V_{,11} V_{\bar{1}}^2) + V_{11} V_{\bar{1}\bar{1}}], \\
I_2 &:= [V_{1\bar{1}}^2 + \alpha^2 \kappa W^\alpha (V_1 V_{\bar{1}})^2 + 2(\alpha \kappa W^\alpha) V_{1\bar{1}} V_1 V_{\bar{1}}].
\end{aligned}$$

It is easy to check that

$$\begin{aligned}
I_1 &= |\Pi V_1^2 + V_{11}|^2, \\
I_2 &= \left[\alpha^2 \kappa (1 - 2\kappa W^\alpha) W^\alpha (V_1 V_{\bar{1}})^2 + \frac{V_{1\bar{1}}^2}{2} + \left| \frac{1}{\sqrt{2}} V_{1\bar{1}} + \sqrt{2} \alpha \kappa W^\alpha V_1 V_{\bar{1}} \right|^2 \right].
\end{aligned}$$

The lemma follows. ■

5.2 Differential inequality-II

Fix a function $g \in \mathcal{R}^\infty(\Omega)$. Denote by $\dot{R}_{i\bar{j}k\bar{l}}$ and $\dot{R}_{i\bar{j}}$ the curvature tensor and the Ricci curvature of (Ω, ω_g) , respectively. Put

$$\dot{\mathcal{R}} := \sqrt{\sum g^{m\bar{n}} g^{k\bar{l}} g^{i\bar{j}} g^{s\bar{t}} \dot{R}_{m\bar{l}i\bar{t}} \dot{R}_{k\bar{n}s\bar{j}}}.$$

Let $f \in \mathcal{R}^\infty(\Omega)$ and $\phi = f - g$. Now we prove a differential inequality of $n - \Box\phi$. Obviously $n - \Box\phi = \sum f^{i\bar{j}} g_{i\bar{j}}$.

Lemma 5.2 (Inequality-II)

$$\Box \log(n - \Box\phi) \geq -\|Ric\|_f - \dot{\mathcal{R}}(n - \Box\phi). \quad (5.9)$$

To prove this lemma we calculate $\Box(n - \Box\phi)$ firstly. In the following calculation, " ; " denotes the covariant derivatives with respect to the metric ω_g .

Lemma 5.3

$$\begin{aligned}
\Box(n - \Box\phi) &\geq \sum f^{i\bar{j}} f^{n\bar{l}} f^{k\bar{h}} f^{m\bar{p}} g_{k\bar{l}} \phi_{;n\bar{p}\bar{j}} \phi_{;m\bar{h}i} - \sum f^{m\bar{l}} f^{k\bar{h}} g_{k\bar{l}} V_{m\bar{h}} \\
&\quad - \dot{\mathcal{R}}(n - \Box\phi)^2.
\end{aligned}$$

Proof. A direct calculation give us

$$f_{;i\bar{j}k} = (g + \phi)_{;i\bar{j}k} = \phi_{;i\bar{j}k}, \quad f_{;i\bar{j}\bar{k}} = \phi_{;i\bar{j}\bar{k}}, \quad f_{;i\bar{j}s\bar{k}} = \phi_{;i\bar{j}s\bar{k}}$$

Note that

$$\sum f_{i\bar{j}} f^{k\bar{j}} = \delta_{ik}$$

Then we have

$$f_{;i}^{k\bar{l}} = - \sum f^{m\bar{l}} f^{k\bar{h}} \phi_{;m\bar{h}i}, \quad f_{;\bar{j}}^{k\bar{l}} = - \sum f^{m\bar{l}} f^{k\bar{h}} \phi_{;m\bar{h}\bar{j}}. \quad (5.10)$$

Then

$$\begin{aligned} (n - \square\phi)_{;i} &= - \sum f^{m\bar{l}} f^{k\bar{h}} \phi_{;m\bar{h}i} g_{k\bar{l}}, \\ \square(n - \square\phi) &= \sum f^{i\bar{j}} g_{k\bar{l}} \left(f^{n\bar{l}} f^{k\bar{h}} f^{m\bar{p}} \phi_{;n\bar{p}\bar{j}} \phi_{;m\bar{h}i} + f^{m\bar{l}} f^{n\bar{h}} f^{k\bar{p}} \phi_{;n\bar{p}\bar{j}} \phi_{;m\bar{h}i} \right. \\ &\quad \left. - f^{m\bar{l}} f^{k\bar{h}} \phi_{;m\bar{h}i\bar{j}} \right), \end{aligned} \quad (5.11)$$

Now we calculate $\phi_{;m\bar{h}i\bar{j}}$. Differentiating the following equation twice

$$-\log H = \log \det (g_{i\bar{j}} + \phi_{i\bar{j}}) - \log \det (g_{i\bar{j}})$$

we have

$$\sum f^{i\bar{j}} \phi_{;i\bar{j}k\bar{l}} = -(\log H)_{k\bar{l}} + \sum f^{i\bar{j}} f^{n\bar{m}} \phi_{;n\bar{j}\bar{l}} \phi_{;\bar{m}ik}. \quad (5.12)$$

By the Ricci identities we have

$$\begin{aligned} \sum f^{i\bar{j}} \phi_{;m\bar{h}i\bar{j}} &= \sum f^{i\bar{j}} f_{;i\bar{h}m\bar{j}} \\ &= \sum f^{i\bar{j}} \left(f_{;i\bar{h}\bar{j}m} + f_{n\bar{h}} \dot{R}_{im\bar{j}}^n - f_{i\bar{n}} \dot{R}_{h\bar{j}m}^{\bar{n}} \right) \\ &= \sum f^{i\bar{j}} \left(f_{;i\bar{j}\bar{h}m} + f_{n\bar{h}} \dot{R}_{im\bar{j}}^n - f_{i\bar{n}} \dot{R}_{h\bar{j}m}^{\bar{n}} \right) \\ &= \sum f^{i\bar{j}} \left(\phi_{;i\bar{j}m\bar{h}} - f_{n\bar{j}} \dot{R}_{im\bar{h}}^n + f_{n\bar{h}} \dot{R}_{im\bar{j}}^n \right) \\ &= V_{;m\bar{h}} + \sum f^{i\bar{j}} f_{n\bar{h}} \dot{R}_{im\bar{j}}^n + \sum f^{i\bar{j}} f^{a\bar{b}} \phi_{;a\bar{j}\bar{h}} \phi_{;\bar{b}im} \end{aligned} \quad (5.13)$$

where we use (5.12) in the last step. Inserting (5.13) into (5.11), we obtain

$$\begin{aligned} \square(n - \square\phi) &\geq \sum f^{i\bar{j}} f^{n\bar{l}} f^{k\bar{h}} f^{m\bar{p}} g_{k\bar{l}} \phi_{;n\bar{p}\bar{j}} \phi_{;m\bar{h}i} - \sum f^{m\bar{l}} f^{k\bar{h}} g_{k\bar{l}} V_{m\bar{h}} \\ &\quad - \sum f^{i\bar{j}} f^{m\bar{l}} \dot{R}_{i\bar{l}m\bar{j}}. \end{aligned} \quad (5.14)$$

To obtain lemma we only need to prove

$$\sum f^{i\bar{j}} f^{m\bar{l}} \dot{R}_{i\bar{l}m\bar{j}} \leq \dot{\mathcal{R}} (n - \square\phi)^2. \quad (5.15)$$

Note that this inequality is invariant under coordinate transformations. We choose another coordinate system so that $g_{k\bar{l}} = \delta_{kl}$, $f_{k\bar{l}} = \delta_{kl} f_{kk}$. Then

$$n - \square\phi = \sum f^{k\bar{k}}, \quad |\dot{R}_{i\bar{l}m\bar{j}}| \leq \dot{\mathcal{R}}.$$

A direct calculation gives us (5.15). ■

Proof of Lemma 5.9. We choose another coordinate system so that $g_{k\bar{l}} = \delta_{kl}$, $f_{k\bar{l}} = \delta_{kl}(1 + \phi_{k\bar{k}})$. Then $n - \square\phi = \sum(1 + \phi_{i\bar{i}})^{-1}$. By the Cauchy inequality one can show that

$$\frac{\left| \sum f^{m\bar{l}} f^{k\bar{h}} g_{k\bar{l}} V_{m\bar{h}} \right|}{n - \square\phi} = \frac{\sum (1 + \phi_{k\bar{k}})^{-2} V_{k\bar{k}}}{\sum (1 + \phi_{i\bar{i}})^{-1}} \leq \|V_{m\bar{l}}\|_f. \quad (5.16)$$

Now, by using a similar calculation as in [56], we prove the following inequality:

$$\sum f^{i\bar{j}} f^{n\bar{l}} f^{k\bar{h}} f^{m\bar{p}} g_{k\bar{l}} \phi_{;n\bar{p}\bar{j}} \phi_{;m\bar{h}i} \geq \frac{\sum f^{i\bar{j}} (\square\phi)_i (\square\phi)_{\bar{j}}}{n - \square\phi}. \quad (5.17)$$

A direct calculation gives us

$$(\square\phi)_i = (n - \sum f^{k\bar{l}} g_{k\bar{l}})_{;i} = \sum f^{k\bar{a}} f^{b\bar{l}} g_{k\bar{l}} \phi_{;a\bar{b}i} = \sum (1 + \phi_{k\bar{k}})^{-2} \phi_{;k\bar{k}i}$$

and

$$\begin{aligned} & \frac{\sum f^{i\bar{j}} (\square\phi)_i (\square\phi)_{\bar{j}}}{n - \square\phi} = \\ &= (n - \square\phi)^{-1} \sum_i (1 + \phi_{i\bar{i}})^{-1} \left| \sum_k (1 + \phi_{k\bar{k}})^{-2} \phi_{;k\bar{k}i} \right|^2 \\ &= (n - \square\phi)^{-1} \sum_i (1 + \phi_{i\bar{i}})^{-1} \left| \sum_k (1 + \phi_{k\bar{k}})^{-\frac{3}{2}} \phi_{;k\bar{k}i} (1 + \phi_{k\bar{k}})^{-\frac{1}{2}} \right|^2 \\ &\leq (n - \square\phi)^{-1} \left(\sum_{i,k} (1 + \phi_{i\bar{i}})^{-1} (1 + \phi_{k\bar{k}})^{-3} \phi_{;k\bar{k}i} \phi_{;k\bar{k}\bar{i}} \right) \left(\sum_k (1 + \phi_{k\bar{k}})^{-1} \right) \\ &= \sum_{i,k} (1 + \phi_{i\bar{i}})^{-1} (1 + \phi_{k\bar{k}})^{-3} \phi_{;k\bar{k}i} \phi_{;k\bar{k}\bar{i}} \\ &\leq \sum_{i,k,l} (1 + \phi_{i\bar{i}})^{-1} (1 + \phi_{k\bar{k}})^{-1} (1 + \phi_{l\bar{l}})^{-2} \phi_{;k\bar{l}i} \phi_{;l\bar{k}\bar{i}} \\ &= \sum f^{i\bar{j}} f^{n\bar{l}} f^{k\bar{h}} f^{m\bar{p}} g_{k\bar{l}} \phi_{;n\bar{p}\bar{j}} \phi_{;m\bar{h}i} \end{aligned}$$

Thus (5.17) is proved. Note that

$$\square \log(n - \square\phi) = \frac{\square(n - \square\phi)}{n - \square\phi} - \frac{\sum f^{i\bar{j}} (n - \square\phi)_i (n - \square\phi)_{\bar{j}}}{(n - \square\phi)^2}.$$

Using (5.16), (5.17) and Lemma 5.3, we obtain Lemma 5.2. ■

5.3 Differential Inequality-III

In this subsection, we apply Differential Inequality-I on the toric surfaces. We use notations introduced in §1.1 and §1.2. Let $0 \in \Omega \subset \mathbb{C}_{\vartheta}^2$, $f = f_{\vartheta}$ be the potential function on Ω . Let $T = \sum f^{i\bar{i}}$. Put

$$Q = e^{N_1(|z|^2 - A)} \sqrt{WT},$$

where A, N_1 are constants.

Lemma 5.4 *Let $K = \mathcal{S}(f) \circ \tau_f^{-1}$ be the scalar curvature function on \mathfrak{t} . Suppose that on Ω ,*

$$\max_{\Omega} \left(|K| + \sum \left| \frac{\partial K}{\partial \xi_i} \right| \right) \leq \mathbf{N}_2, \quad W \leq \mathbf{N}_2, \quad |z| \leq \mathbf{N}_2 \quad (5.18)$$

for some constant $\mathbf{N}_2 > 0$. Then we may choose

$$A = \mathbf{N}_2^2 + 1, N_1 = 100, \alpha = \frac{1}{3}, \kappa = [4\mathbf{N}_2^{\frac{1}{3}}]^{-1} \quad (5.19)$$

such that

$$\square(P + Q + \mathbf{C}_7 f) \geq \mathbf{C}_8 (P + Q)^2 > 0 \quad (5.20)$$

for some positive constants \mathbf{C}_7 and \mathbf{C}_8 that depend only on \mathbf{N}_2 and n .

Proof. Applying Lemma 5.1 and the choice of α and κ , in particular, $\kappa W^\alpha \leq 1/4$, we have

$$\frac{\Psi \square P}{P} \geq \left(\frac{1}{2} \|V_{,i\bar{j}}\|_f^2 + \frac{1}{18} \kappa W^{\frac{1}{3}} \Psi^2 \right) - (2|\langle \nabla \mathcal{S}, \nabla V \rangle| + \Psi |\mathcal{S}|). \quad (5.21)$$

Treatment for $\langle \nabla \mathcal{S}, \nabla \log W \rangle$: using log-affine coordinates we have

$$|\langle \nabla \mathcal{S}, \nabla V \rangle| = \left| \sum f^{ij} \frac{\partial \mathcal{S}}{\partial x_i} \frac{\partial V}{\partial x_j} \right| = \left| \sum f^{ij} \frac{\partial K}{\partial \xi_k} \frac{\partial \xi_k}{\partial x_i} \frac{\partial V}{\partial x_j} \right| \leq \mathbf{N}_2 \sum_j \left| \frac{\partial V}{\partial x_j} \right|,$$

where we use the fact $\frac{\partial \xi_k}{\partial x_i} = f_{ki}$; now we back to the complex coordinates z_i , we have

$$\left| \frac{\partial V}{\partial x_j} \right| = |z_j V_j|.$$

Since $|z|$ is bounded, we conclude that

$$|\langle \nabla \mathcal{S}, \nabla V \rangle| \leq C \sum_j |V_j| \leq C \sqrt{2WT\Psi}.$$

We explain the last step: suppose that $0 < \nu_1 \leq \nu_2$ are the eigenvalues of $(f_{i\bar{j}})$, then

$$\Psi = \sum f^{i\bar{j}} V_i V_{\bar{j}} \geq \nu_2^{-1} (|V_1|^2 + |V_2|^2) \geq (WT)^{-1} (|V_1|^2 + |V_2|^2).$$

Note that

$$(eW)^{-\frac{1}{2}} \leq \Psi/P = (\exp(\kappa W^\alpha)W^{\frac{1}{2}})^{-1} \leq W^{-\frac{1}{2}},$$

(5.21) is then transformed to be

$$\square P \geq W^{\frac{1}{2}} \left(\frac{1}{2} \|V_{,i\bar{j}}\|_f^2 + \frac{1}{18} \kappa W^{\frac{1}{3}} \Psi^2 \right) - C' W^{\frac{1}{2}} \left(\sqrt{WT\Psi} + \Psi|\mathcal{S}| \right). \quad (5.22)$$

Applying the Young inequality and the Schwartz inequality to terms in (5.22), we have that

$$\square P \geq \frac{1}{2} W^{\frac{1}{2}} \|V_{,i\bar{j}}\|_f^2 + C_1 W^{-\frac{1}{6}} P^2 - \epsilon QT - C_2(\epsilon). \quad (5.23)$$

For example,

$$W(T\Psi)^{\frac{1}{2}} \leq \delta(W^{\frac{1}{8}} T^{\frac{1}{2}})^4 + \delta(W^{\frac{5}{24}} \Psi^{\frac{1}{2}})^4 + C_\delta W^{\frac{4}{3}} \leq C'_2 \delta(QT + W^{-\frac{1}{6}} P^2) + C'_\delta.$$

We skip the proof of (5.23).

By a direct calculation we have

$$\square Q \geq Q \left(N_1 T + \frac{1}{2} \square V + \square \log T \right) \geq Q \left(N_1 T - \frac{1}{2} \mathcal{S} - \|V_{,i\bar{j}}\|_f \right),$$

where we use the formula (5.9) with $g_{ij} = \delta_{ij}$ to calculate $\square \log T$. Using the explicit value A, N_1 and the bounds of W and \mathcal{S} , applying the Schwartz inequality properly, we can get

$$\square Q \geq -\frac{1}{4} W^{\frac{1}{2}} \|V_{,i\bar{j}}\|_f^2 + \frac{N_1}{3} QT - C_3(N_1, N_2). \quad (5.24)$$

Combining (5.23) and (5.24), choosing $\epsilon = \frac{1}{100}$, we have

$$\square(Q + P) \geq C_1 W^{-\frac{1}{6}} P^2 + \frac{N_1}{4} QT - C_4.$$

Note that

$$T = e^{-N_1(|z|^2 - A)} W^{-\frac{1}{2}} Q \geq e^{-N_1(|z|^2 - A)} N_2^{-\frac{1}{3}} W^{-\frac{1}{6}} Q \geq C_5 W^{-\frac{1}{6}} Q,$$

we get

$$\square(Q + P) \geq C_6 W^{-\frac{1}{6}} (Q + P)^2 - C_7$$

where C_6, C_7 are constants depending only on C_1, N_1 and N_2 . Our lemma follows from $\square f = n$ and $|W| \leq N_2$. ■

6 Complex interior estimates and regularities

We apply the differential inequalities to derive interior estimates in terms of the norm \mathcal{K} of Ricci tensor (cf. (1.3)).

6.1 Interior estimate of Ψ

We use Lemma 5.1 to derive the interior estimate of Ψ in a geodesic ball.

Lemma 6.1 *Let $f \in \mathcal{R}^\infty(\Omega)$ and $B_a(o) \subset \Omega$ be the geodesic ball of radius a centered at o . Suppose that*

$$W \leq 1, \quad W^{\frac{1}{2}}(o')\mathcal{K} \leq \mathbf{N}_3, \quad (6.1)$$

in $B_a(o)$, where $o' \in B_a(o)$. Then the following estimate holds in $B_{a/2}(o)$

$$W^{\frac{1}{2}}\Psi \leq C_6 \left[\max_{B_a(o)} \left(|S| + \|\nabla S\|_f^{\frac{2}{3}} \right) + a^{-1}[W(o')]^{-\frac{1}{4}} + a^{-2} \right] \quad (6.2)$$

where C_6 is a constant depending only on n and \mathbf{N}_3 .

Proof. Consider the function

$$F := (a^2 - r^2)^2 P$$

defined in $B_a(o)$, where r denotes the geodesic distance from o to z with respect to the metric ω_f . F attains its supremum at some interior point p^* . Then, at p^* ,

$$\begin{aligned} \frac{P_{,i}}{P} - \frac{2(r^2)_{,i}}{(a^2 - r^2)} &= 0, \\ \frac{\square P}{P} - \frac{\|\nabla P\|_f^2}{P^2} - \frac{2\|\nabla(r^2)\|_f^2}{(a^2 - r^2)^2} - \frac{2\square(r^2)}{(a^2 - r^2)} &\leq 0. \end{aligned}$$

Substituting the first equation into the second inequality we have

$$\frac{\square P}{P} - \frac{6\|\nabla(r^2)\|_f^2}{(a^2 - r^2)^2} - \frac{2\square(r^2)}{(a^2 - r^2)} \leq 0. \quad (6.3)$$

In the following computation, we remind that we will use the fact $W \leq 1$ frequently. On the one hand, by Inequality-I(Lemma 5.1) and choose

$$\kappa = \frac{1}{8}, \alpha = \frac{1}{3}. \quad (6.4)$$

we have

$$\frac{\square P}{P} \geq \frac{1}{2}\alpha^2 \kappa W^\alpha \Psi - \left[\frac{2\|\nabla S\|_f}{\sqrt{\Psi}} + \left(\alpha \kappa W^\alpha + \frac{1}{2} \right) S \right]. \quad (6.5)$$

On the other hand, since $\mathcal{K} \leq C_1 W^{-\frac{1}{2}}(o')$, we have $Ric \geq -nC_1 W^{-\frac{1}{2}}(o')$. By the Laplacian comparison theorem (cf. Remark 6.2) we have

$$r\square r \leq C(n) \left(1 + [W(o')]^{-\frac{1}{4}} a \right). \quad (6.6)$$

Insert (6.5) and (6.6) into (6.3), then,

$$W^\alpha \Psi \leq C_4 \left[\frac{a[W(o')]^{-\frac{1}{4}}}{a^2 - r^2} + \frac{a^2}{(a^2 - r^2)^2} + |\mathcal{S}| + \frac{\sqrt{W^\alpha} \|\nabla \mathcal{S}\|_f}{\sqrt{W^\alpha \Psi}} \right],$$

for some constant $C_4 > 0$. Multiplying $[W^\alpha \Psi]^{\frac{1}{2}}$ on both sides, and applying Young's inequality, we get

$$[W^\alpha \Psi]^{\frac{3}{2}} \leq C'_4 \left[\left(\frac{a[W(o')]^{-\frac{1}{4}}}{(a^2 - r^2)} + \frac{a^2}{(a^2 - r^2)^2} + |\mathcal{S}| \right)^{\frac{3}{2}} + \|\nabla \mathcal{S}\|_f \right].$$

It follows that

$$W^{\frac{1}{2}} \Psi \leq C_5 \left[\frac{a[W(o')]^{-\frac{1}{4}}}{a^2 - r^2} + \frac{a^2}{(a^2 - r^2)^2} + |\mathcal{S}| + \|\nabla \mathcal{S}\|_f^{\frac{2}{3}} \right].$$

Then

$$F(p^*) \leq C'_5 \left[a^3 [W(o')]^{-\frac{1}{4}} + a^2 + a^4 \max_{B_a(o)} \left(|\mathcal{S}| + \|\nabla \mathcal{S}\|_f^{\frac{2}{3}} \right) \right].$$

Since $F \leq F(p^*)$, we obtain

$$(a^2 - r^2)^2 P \leq C'_5 \left[a^3 [W(o')]^{-\frac{1}{4}} + a^2 + a^4 \max_{B_a(o)} \left(|\mathcal{S}| + \|\nabla \mathcal{S}\|_f^{\frac{2}{3}} \right) \right].$$

Note that

$$(a^2 - r^2)^2 P = (a^2 - r^2)^2 \exp(\kappa W^\alpha) W^{\frac{1}{2}} \Psi \geq (a^2 - r^2)^2 W^{\frac{1}{2}} \Psi.$$

We can obtain the lemma easily. ■

Remark 6.2 (The Laplacian comparison theorem) *Let M^n be complete Riemannian manifold with $\text{Ric} \geq -(n-1)C_0$, where $C_0 > 0$. Then the geodesic distance function r from p satisfies*

$$r \square r(x) \leq C(n)(1 + \sqrt{C_0} r(x)).$$

By the same argument of Lemma 6.1, we have

Lemma 6.3 *Let $f \in \mathcal{R}^\infty(\Omega)$ and $B_a(o) \subset \Omega$ be the geodesic ball of radius a centered at o . Suppose that there are constants $N_1, N_2 > 0$ such that*

$$\mathcal{K} \leq N_1, \quad W \leq N_2,$$

in $B_a(o)$. Then in $B_{a/2}(o)$

$$W^{\frac{1}{2}} \Psi \leq C_9 N_2^{\frac{1}{2}} \left[\max_{B_a(o)} \left(|\mathcal{S}| + \|\nabla \mathcal{S}\|_f^{\frac{2}{3}} \right) + a^{-1} + a^{-2} \right].$$

where C_9 is a constant depending only on n and N_1 .

Remark 6.4 Recall that $\Psi = \|\nabla \log W\|_f^2$. Hence

$$W^{\frac{1}{2}}\Psi = 16\|\nabla W^{\frac{1}{4}}\|_f^2.$$

Therefore, the result in Lemma 6.1 and Lemma 6.3 can be treated as a bound of $\nabla W^{\frac{1}{4}}$.

6.2 Interior estimate of $\|\nabla f\|_f$

Lemma 6.5 Let $f \in \mathcal{R}^\infty(\Omega)$ with $f(p_0) = \inf_\Omega f = 0$. Suppose that for $a > 1$,

$$\mathcal{K} \leq N_0, \text{ in } B_a(p_0). \quad (6.7)$$

Then in $B_{a/2}(p_0)$

$$\frac{\|\nabla f\|_f^2}{(1+f)^2} \leq C_{10} \quad (6.8)$$

where $C_{10} > 0$ is a constant depending only on n and N_0 . Then, for any $q \in B_{a/2}(p_0)$,

$$f(q) - f(p_0) \leq \exp(\sqrt{C_{10}a}). \quad (6.9)$$

Proof. Consider the function

$$F = (a^2 - r^2)^2 \frac{\sum f^{i\bar{j}} f_i f_{\bar{j}}}{(1+f)^2},$$

in $B_a(p_0)$. F attains its supremum at some interior point p^* . Then, at p^* ,

$$-\left[\frac{2(r^2)_{,k}}{a^2 - r^2} + 2\frac{f_{,k}}{1+f} \right] \sum f^{i\bar{j}} f_i f_{\bar{j}} + \sum f^{i\bar{j}} f_{,ik} f_{\bar{j}} + f_k = 0, \quad (6.10)$$

$$\begin{aligned} & -\left[\frac{2\|\nabla r^2\|_f^2}{(a^2 - r^2)^2} + \frac{2\Box(r^2)}{a^2 - r^2} + \frac{2\Box f}{1+f} \right] \|\nabla f\|_f^2 + n + \sum f^{i\bar{j}} f^{k\bar{l}} f_{,ki} f_{,\bar{l}\bar{j}} + \sum f^{i\bar{j}} f^{k\bar{l}} f_{,ik\bar{l}} f_{\bar{j}} \\ & + \frac{2\|\nabla f\|_f^4}{(1+f)^2} - \sum \left[\frac{2(r^2)_{,k}}{a^2 - r^2} + 2\frac{f_{,k}}{1+f} \right] f^{k\bar{l}} f^{i\bar{j}} (f_{i\bar{l}} f_{\bar{j}} + f_i f_{,\bar{l}\bar{j}}) \leq 0. \end{aligned} \quad (6.11)$$

Choose the coordinate system such that at p^* , we have

$$f_{i\bar{j}} = \delta_{ij}, \quad f_{,1} = f_{,\bar{1}} = \|\nabla f\|_f, \quad f_{,i} = f_{,\bar{i}} = 0, \quad \text{for } i \geq 2.$$

Then (6.10) and (6.11) can be read as

$$-\left[\frac{2(r^2)_{,k}}{a^2 - r^2} + 2\frac{f_{,k}}{1+f} \right] f_1 f_{\bar{1}} + f_{,1k} f_{\bar{1}} + f_1 \delta_{1k} = 0, \quad (6.12)$$

$$\begin{aligned} & -\left[\frac{2\|\nabla r^2\|_f^2}{(a^2 - r^2)^2} + \frac{2\Box(r^2)}{a^2 - r^2} + \frac{2\Box f}{1+f} \right] f_1 f_{\bar{1}} + n + \sum f_{,lk} f_{,\bar{l}\bar{k}} + \sum f_{,1k\bar{k}} f_{\bar{1}} \\ & + \frac{2(f_1 f_{\bar{1}})^2}{(1+f)^2} - \sum \left[\frac{2(r^2)_{,k}}{a^2 - r^2} + 2\frac{f_{,k}}{1+f} \right] (\delta_{1k} f_{\bar{1}} + f_1 f_{,\bar{1}\bar{k}}) \leq 0. \end{aligned} \quad (6.13)$$

Applying the Ricci identities, the fact $\sum r_k r_{\bar{k}} = \frac{1}{4}$, and the Laplacian comparison Theorem to (6.13), and inserting (6.7) then

$$\begin{aligned} & - \left[\frac{3a^2}{(a^2 - r^2)^2} + \frac{C_1(n)(1 + \sqrt{N_0}a)}{a^2 - r^2} + \frac{2n}{1+f} - N_0 \right] f_1 f_{\bar{1}} + n + \frac{2(f_1 f_{\bar{1}})^2}{(1+f)^2} \\ & + \sum f_{,lk} f_{,\bar{l}\bar{k}} - \left| \sum \left[\frac{2(r^2)_{,k}}{a^2 - r^2} + \frac{2f_{,k}}{1+f} \right] f_1 f_{,\bar{1}\bar{k}} \right| - \left| \frac{2(r^2)_{,1} f_{\bar{1}}}{a^2 - r^2} + \frac{2f_{,1} f_{\bar{1}}}{1+f} \right| \leq 0. \end{aligned} \quad (6.14)$$

Note that

$$\begin{aligned} 2 \left| \sum \left[\frac{(r^2)_{,k}}{a^2 - r^2} + \frac{f_{,1} \delta_{1k}}{1+f} \right] f_1 f_{,\bar{1}\bar{k}} \right| & \leq \sum \left| \frac{(r^2)_{,k}}{a^2 - r^2} + \frac{f_{,1} \delta_{1k}}{1+f} \right|^2 f_{,1} f_{,\bar{1}} + \sum f_{,\bar{1}\bar{k}} f_{,1k} \\ & = \frac{\|\nabla r^2\|_f^2 f_{,1} f_{,\bar{1}}}{(a^2 - r^2)^2} + 2Re \left(\frac{(r^2)_{,1}}{a^2 - r^2} \frac{f_{,1}^2 f_{,\bar{1}}}{1+f} \right) + \frac{(f_{,1} f_{,\bar{1}})^2}{(1+f)^2} + \sum f_{,1k} f_{,\bar{1}\bar{k}}. \end{aligned}$$

Inserting the above inequality into (6.14) we have

$$\begin{aligned} & - \left[\frac{C_2 a^2}{(a^2 - r^2)^2} + \frac{C_3 a}{a^2 - r^2} + \frac{2n}{1+f} \right] f_1 f_{\bar{1}} + \frac{(f_1 f_{\bar{1}})^2}{(1+f)^2} \\ & - N_0 f_1 f_{\bar{1}} - 2 \left| \frac{(r^2)_{,1} f_{\bar{1}}}{a^2 - r^2} \right| + \frac{2f_1 f_{\bar{1}}}{1+f} + 2Re \left[\frac{(r^2)_{,1}}{a^2 - r^2} \frac{f_{,1}^2 f_{,\bar{1}}}{1+f} \right] \leq 0. \end{aligned} \quad (6.15)$$

We discuss two cases:

Case 1. $\frac{f_1 f_{\bar{1}}}{(1+f)^2}(p^*) \leq 1$.

$$F \leq F(p^*) \leq \frac{f_1 f_{\bar{1}}}{(1+f)^2}(p^*) a^4 \leq a^4.$$

Case 2. $\frac{f_1 f_{\bar{1}}}{(1+f)^2}(p^*) > 1$. We apply Schwarz's inequality in (6.15) to get the following inequality

$$\frac{f_1 f_{\bar{1}}}{(1+f)^2} \leq C_6(n) + C_7 \left(\frac{a^2}{(a^2 - r^2)^2} + \frac{a}{a^2 - r^2} \right).$$

Hence

$$F \leq C_6 a^4 + C_7(a^2 + a^3).$$

Then in $B_{\frac{a}{2}}(p_0)$ we obtain (6.8). \blacksquare

6.3 Interior estimate of eigenvalues of $(f_{i\bar{j}})$

For simplicity, we assume that $g = \sum z_i z_{\bar{i}}$. Denote $T = \sum f^{i\bar{i}}$. Then Inequality-II (Lemma 5.2) can be re-written as

$$\square \log T \geq -\|Ric\|_f. \quad (6.16)$$

We apply this to prove the following lemma.

Lemma 6.6 *Let $f \in \mathcal{R}^\infty(\Omega)$ and $B_a(p_0) \subset \Omega$. If*

$$W \leq \mathbf{N}_4, \quad \mathcal{K} \leq \mathbf{N}_4, \quad |z| \leq \mathbf{N}_4.$$

in $B_a(p_0)$, for some constant $\mathbf{N}_4 > 0$. Then there exists a constant $\mathbf{C}_{11} > 1$ such that

$$\mathbf{C}_{11}^{-1} \leq \lambda_1 \leq \cdots \leq \lambda_n \leq \mathbf{C}_{11}, \quad \forall q \in B_{a/2}(p_0).$$

where $\lambda_1, \dots, \lambda_n$ are eigenvalues of the matrix $(f_{i\bar{j}})$, \mathbf{C}_{11} is a positive constant depending on n, a and \mathbf{N}_4 .

Proof. Consider the function

$$F := (a^2 - r^2)^2 e^{|z|^2} T$$

in $B_a(p_0)$. F attains its supremum at some interior point p^* . Then, at p^* , we have

$$\begin{aligned} z_{\bar{i}} + (\log T)_{,i} - \frac{2(r^2)_{,i}}{a^2 - r^2} &= 0, \\ T + \square \log T - \frac{2\|\nabla r^2\|_f^2}{(a^2 - r^2)^2} - \frac{2\square(r^2)}{a^2 - r^2} &\leq 0. \end{aligned} \quad (6.17)$$

Note that $\mathcal{K} \leq \mathbf{N}_4$. Thus, by (6.16) and the Laplacian comparison theorem, we have

$$T \leq \frac{b_1 a^2}{(a^2 - r^2)^2} + \frac{b_2 a}{a^2 - r^2} + \mathbf{N}_4,$$

for some positive constants b_1, b_2 depending only on n and \mathbf{N}_4 . Then

$$F \leq C_3 a^3 + C_4 a^4,$$

for some positive constants C_3 and C_4 that depends only on n and \mathbf{N}_4 . Then there is a constant $C_5 > 0$ such that

$$\lambda_1^{-1} \leq \sum f^{i\bar{i}} \leq C_5 \quad \text{in } B_{\frac{a}{2}}(p_0, \omega_f).$$

Since W is bounded and λ_1 is bounded below, hence λ_n is bounded above. ■

6.4 Bootstrapping

In the subsection, for the sake of convenience, we state and prove our results on the toric surfaces. Suppose that $\mathcal{S}(u) = K$ and $K \in C^\infty(\bar{\Delta})$. Then on the complex manifold, the equation is $\mathcal{S}(f) = \tilde{K}$. For example, on the complex torus in terms of log-affine coordinates:

$$\tilde{K} = K \circ \nabla^f.$$

The bootstrapping argument says that the regularity of f can be obtained via $\|\tilde{K}\|_{C^\infty}$ if $\|f\|_{C^2}$ is bounded. However, it is not so obvious that this is true in terms of $\|K\|_{C^\infty}$. In this subsection, we verify this by direct computations. The main theorem in this subsection is

Theorem 6.7 *Let U to be one of the coordinate charts U_Δ, U_ℓ and U_ϑ . Suppose that $U \subset U$ is a bounded set and*

$$C_1^{-1} \leq \nu_1 \leq \nu_2 \leq C_1 \quad (6.18)$$

for some constant $C_1 > 0$, where ν_1, ν_2 are the eigenvalues of the matrix $(\sum g^{i\bar{j}} f_{k\bar{j}})$. Then for any $U' \subset U$ and any k

$$\|f\|_{C^k(U')} \leq C_k,$$

where C_k depends on $\|K\|_{C^\infty}$, $d_E(U', \partial U)$ and the bound of U .

The rest of subsection is the proof of the theorem.

Case 1, bootstrapping on U_Δ . The interior regularity of u in Δ is equivalent to that of f in the complex torus. Hence we may apply the bootstrapping argument for u to conclude the regularity of f .

Case 2, bootstrapping on U_ℓ . Without loss of generality, we assume that $U_\ell = U_{h^*}$. One of the main issue is to study the relation between the derivatives of $S(f)$ and $S(u)$. Recall that the complex coordinate is (z_1, w_2) . By the explicit computation, we have

$$\tilde{K}(z_1, w_2) = K(\xi_1, \xi_2),$$

where

$$\xi_1 = z_1 \frac{\partial f}{\partial z_1}, \quad \xi_2 = 2 \frac{\partial f}{\partial w_2}.$$

Then by the direct computation, we have

Lemma 6.8 *For any k there exists a constant C'_k such that*

$$\|\tilde{K}\|_{C^k(U)} \leq C'_k,$$

where C'_k depends only on $\|f\|_{C^{k+1}(U)}$, $|K|_{C^k}$ and the bound of U .

Proof. When $k = 1$, then

$$\frac{\partial \tilde{K}}{\partial z_1} = \frac{\partial K}{\partial \xi_1} \left(\frac{\partial f}{\partial z_1} + z_1 \frac{\partial^2 f}{\partial^2 z_1} \right) + 2 \frac{\partial K}{\partial \xi_2} \frac{\partial^2 f}{\partial z_1 \partial w_2}.$$

Other derivatives $\partial \tilde{K} / \partial \bar{z}_1$, $\partial \tilde{K} / \partial w_2$ and $\partial \tilde{K} / \partial \bar{w}_2$ can be computed similarly. We verify the assertion for $k = 1$. It is easy to see that the assertion is true for any k . q.e.d.

When $\|f\|_{C^2(U)} \leq C$ for some constant $C > 0$, applying Lemma 6.8 with $k = 1$, we have $\|\tilde{K}\|_{C^1(U)} \leq C'$. Since $C_0^{-1} \leq (g_{i\bar{j}}) \leq C_0$ in U for some constant $C_0 > 0$, by (6.18) we get $C_1^{-1} \leq (f_{i\bar{j}}) \leq C_1$ for some constant $C_1 > 0$. Applying the L^p estimates to the equation $\square \log W = -K$, we have for any $p > 2$, $W \in W^{2,p}(U)$. It follows from the Sobolev embedding theorem that $W \in C^{1,\alpha}(U)$. Using Krylov-Evans Lemma and bootstrapping argument to the equation $S(f) = \tilde{K}$, we have $\|f\|_{C^{4,\alpha}(U)} \leq C'_1$. Hence, it is crucial to get the interior estimate of $\|f\|_{C^2}$. We need the following lemma.

Lemma 6.9 *Let ν_1, ν_2 be the eigenvalues of the matrix $(\sum g^{i\bar{j}} f_{k\bar{j}})$. Suppose that in U*

$$C_1^{-1} \leq \nu_1 \leq \nu_2 \leq C_1, \quad \|D^2 g\|_{C^0(U)} \leq C_1 \quad (6.19)$$

for some constant $C_1 > 0$. Then there is a constant $C_2 > 0$ such that

$$\|D^2 f\|_{C^0(U)} \leq C_2.$$

Proof. In log-coordinate (w_1, w_2) we have

$$\frac{\partial^2 f}{\partial w_i \partial w_j} = \frac{\partial^2 f}{\partial w_i \partial \bar{w}_j} = \frac{\partial^2 f}{\partial \bar{w}_i \partial \bar{w}_j},$$

and

$$\lim_{x_1 \rightarrow -\infty} \frac{\partial g}{\partial x_1}(x) = \lim_{x_1 \rightarrow -\infty} \frac{\partial f}{\partial x_1}(x) = 0. \quad (6.20)$$

It follows from (6.19) and (6.20) that

$$0 < \frac{\partial}{\partial x_1} f \leq C_1 \frac{\partial}{\partial x_1} g. \quad (6.21)$$

Obviously, (6.19) gives us

$$\left| \frac{\partial^2 f}{\partial w_2 \partial \bar{w}_2} \right| + \left| \frac{\partial^2 f}{\partial z_1 \partial \bar{z}_1} \right| + \left| \frac{\partial^2 f}{\partial z_1 \partial \bar{w}_2} \right| + \left| \frac{\partial^2 f}{\partial w_2 \partial \bar{z}_1} \right| \leq C_3 \quad (6.22)$$

for some constant $C_3 > 0$. Since $\frac{\partial f}{\partial \bar{w}_2} = \frac{\partial f}{\partial w_2}$, we have

$$\left| \frac{\partial^2 f}{\partial w_2 \partial w_2} \right| + \left| \frac{\partial^2 f}{\partial z_1 \partial w_2} \right| \leq C_3.$$

Note that

$$\frac{\partial^2 f}{\partial z_1^2} = \sum \frac{\partial^2 f}{\partial w_1^2} \left(\frac{\partial w_1}{\partial z_1} \right)^2 + \frac{\partial^2 w_1}{\partial z_1^2} \frac{\partial f}{\partial w_1}.$$

Using (6.22), we have

$$\left| \frac{\partial^2 f}{\partial w_1^2} \left(\frac{\partial w_1}{\partial z_1} \right)^2 \right| = \left| \frac{\partial^2 f}{\partial w_1 \partial \bar{w}_1} \frac{\partial w_1}{\partial z_1} \frac{\partial \bar{w}_1}{\partial \bar{z}_1} \right| = \left| \frac{\partial^2 f}{\partial z_1 \partial \bar{z}_1} \right| \leq C_3,$$

and by (6.21), we have

$$\begin{aligned} \left| \frac{\partial^2 w_1}{\partial z_1^2} \frac{\partial f}{\partial w_1} \right| &\leq C_2 \left| \frac{\partial^2 w_1}{\partial z_1^2} \frac{\partial g}{\partial w_1} \right| \\ &= C_2 \left| \frac{\partial^2 w_1}{\partial z_1^2} \frac{\partial g}{\partial w_1} + \sum \frac{\partial^2 g}{\partial w_1 \partial w_1} \frac{\partial w_1}{\partial z_1} \frac{\partial w_1}{\partial z_1} - \sum \frac{\partial^2 g}{\partial w_1 \partial w_1} \frac{\partial w_1}{\partial z_1} \frac{\partial w_1}{\partial \bar{z}_1} \right| \\ &\leq C_2 \left| \frac{\partial^2 g}{\partial z_1^2} \right| + C_2 \left| \frac{\partial^2 g}{\partial z_1 \partial \bar{z}_1} \right|. \end{aligned}$$

Then $\|D^2g\|_{C^2(U)} \leq C$ implies that

$$\|D^2f\|_{C^2(U)} \leq C.$$

The lemma is proved. ■

Combining the above two lemmas and the standard bootstrapping argument, we prove the theorem for Case 2.

Case 3, bootstrapping on U_ϑ . The proof is same as that of Case 2.

We complete the proof of Theorem 6.7.

By subtracting a linear function from f we may normalize f such that $f(z) \geq f(z_o) = 0$, this can be easily done if $U = U_\Delta$, when $U = U_\ell$, this is explain in the beginning of §7.1, when $U = U_\vartheta$, this can be done similarly. We leave the verification to readers.

Combining Lemma 6.6 and Theorem 6.7 we have the following theorem.

Theorem 6.10 *Let $z_o \in U$ and $B_a(z_o)$ be a geodesic ball in U . Suppose that there is a constant $C_1 > 0$ such that $f(z_o) = 0$, $\nabla f(z_o) = 0$, and*

$$\mathcal{K}(f) \leq C_1, \quad C_1^{-1} \leq W \leq C_1, \quad |z| \leq C_1$$

in $B_a(z_o)$. Then there is a constant $a_1 > 0$, depending on a and C_1 , such that $D_{2a_1}(z_o) \subset B_{\frac{a}{2}}(z_o)$, and

$$\|f\|_{C^{3,\alpha}(D_{a_1}(z_o))} \leq C(a, C_1, |\mathcal{S}(f)|),$$

$$\|f\|_{C^\infty(D_{a_1}(z_o))} \leq C(a, C_1, \|\mathcal{S}(u)\|_{C^\infty}).$$

Proof. By Lemma 6.6, we conclude that

$$C_2^{-1} \leq \lambda_1 \leq \lambda_2 \leq C_2, \quad \text{in } B_{\frac{a}{2}}(z_o)$$

for some constant $C_2 > 0$, hence, by Lemma 6.9, we have the bounds of $|D^2f|$. Then there is a constant $a_1 > 0$ such that $D_{2a_1}(z_o) \subset B_{\frac{a}{2}}(z_o)$. And by the integrations we have the bound of the norm of $f \in C^1(D_{2a_1}(z_o))$. Combine together, we have the bound of C^2 -norm of f in $D_{2a_1}(z_o)$. Then by applying the Krylov-Evans Lemma, the Schauder estimate and standard bootstrap we have the bound of the norm of $f \in C^\infty(D_{a_1}(z_o))$. ■

Remark 6.11 *The results in this subsection can be easily generalized to high dimension.*

7 Estimates of \mathcal{K} near divisors

Let $\Delta \subset \mathbb{R}^2$. In this section, we prove the following theorem.

Theorem 7.1 *Let $u \in \mathcal{C}^\infty(\Delta, v)$. Let \mathfrak{z}_o be a point on a divisor Z_ℓ for some ℓ . Choose a coordinate system (ξ_1, ξ_2) such that $\ell = \{\xi| \xi_1 = 0\}$. Let $p \in \ell$ and $D_b(p) \cap \bar{\Delta}$ be an Euclidean half-disk such that its intersects with $\partial\Delta$ lies in the interior of ℓ . Let $B_a(\mathfrak{z}_o)$ be a geodesic ball satisfying $\tau_f(B_a(\mathfrak{z}_o)) \subset D_b(p)$. Suppose that in $D_b(p) \cap \Delta$*

$$|\mathcal{S}(u)| \geq \delta > 0, \quad \|\mathcal{S}(u)\|_{C^3(\bar{\Delta})} \leq \mathbf{N}_5, \quad h_{22}|_\ell \geq \mathbf{N}_5^{-1} \quad (7.1)$$

for some constant $\mathbf{N}_5 > 0$, where $h = u|_\ell$ and $\|\cdot\|_{C^3(\bar{\Delta})}$ denotes the Euclidean C^3 -norm. Then there is a constant $\mathbf{C}_{12} > 0$, depending only on a, δ, \mathbf{N}_5 and $D_b(p)$, such that

$$\frac{\min_{B_a(\mathfrak{z}_o) \cap Z_\ell} W}{\max_{B_a(\mathfrak{z}_o)} W} (\mathcal{K}(\mathfrak{z}) + \|\nabla \log |\mathcal{S}|\|^2(\mathfrak{z})) a^2 \leq \mathbf{C}_{12}, \quad \forall \mathfrak{z} \in B_{a/2}(\mathfrak{z}_o) \quad (7.2)$$

where $W = \det(f_{s\bar{t}})$.

7.1 Affine transformation rules on $\mathbb{C} \times \mathbb{C}^*$

Recall that the coordinate chart for ℓ is $\mathcal{U}_\ell \cong \mathbb{C} \times \mathbb{C}^*$. By the assumption, $B_a(\mathfrak{z}_o)$ is inside this chart. The situation is same as in \mathcal{U}_{h^*} . Hence for the sake of simplicity of notations, we assume that we are working at \mathcal{U}_{h^*} . Let $u \in \mathcal{C}^\infty(h^*, v_{h^*})$. Then $f = L(u)$ is a function on \mathfrak{t} . Hence it defines a function on the $\mathbb{C}^* \times \mathbb{C}^* \subset \mathcal{U}_{h^*}$ in terms of log-affine coordinate (w_1, w_2) . Then the function $f_h(z_1, w_2) := f(\log |z_1^2|, \text{Re}(w_2))$ extends smoothly over Z , hence defines on \mathcal{U}_{h^*} . We denote f_h by f to simplify notations. Then

$$\lim_{x_1 \rightarrow -\infty} \frac{\partial f}{\partial x_1} = 0, \quad \frac{\partial f}{\partial x_1} > 0.$$

Fix a point $\mathfrak{z}_o \in Z$, we claim that by subtracting a linear function h on \mathfrak{t} , for $\hat{f} = f - h$, $\hat{f}(\mathfrak{z}_o) = \min \hat{f}$. In fact, let $(0, a) = \tau_f(\mathfrak{z}_o)$. Then

$$\hat{f}(x_1, x_2) = f(x_1, x_2) - ax_2.$$

In terms of complex coordinate,

$$\hat{f}(z_1, w_2) = f(z_1, w_2) - \frac{1}{2}a(w_2 + \bar{w}_2).$$

We find that $\tau_{\hat{f}}(\mathfrak{z}_o) = (0, 0)$. Hence it is easy to show that \hat{f} achieves its minimal at \mathfrak{z}_o .

We will explain how the affine transformation affects the invariants of functions.

Let $u \in \mathcal{C}^\infty(h^*, v_{h^*})$. We consider the following affine transformation on u :

$$\tilde{u}(\xi) = \lambda u(A^{-1}(\xi)) + \eta \xi_1 + b \xi_2 + c, \quad (7.3)$$

where $A(\xi_1, \xi_2) = (\alpha\xi_1, \beta\xi_2 + \gamma)$. Let $\tilde{f} = L(\tilde{u})$. In this section, we always take $\lambda = \alpha$. Then \tilde{f} is still a potential function of U_ℓ . \tilde{f} can be computed directly. The above transformation induces a transformation on \mathfrak{t} :

$$B(x_1, x_2) = (x_1 + \eta, \frac{\lambda}{\beta}x_2 + b).$$

Then by a direct computation, we have

Lemma 7.2 $\tilde{f} = \lambda f \circ B^{-1} + \gamma x_2 - c - \gamma b$.

Now we explain the coordinate change in complex sense that covers B . Let

$$\begin{array}{ccc} \mathbb{R}^1 \times \mathbb{R}^1 & \longrightarrow & \mathbb{R}^1 \times \mathbb{S}^1 \\ (x_2, \check{y}_2) & \xrightarrow{(I, p_r)} & (x_2, y_2). \end{array}$$

be the universal holomorphic covering of \mathbb{C}^* . It induces a covering map

$$\begin{array}{ccc} \mathbb{C} \times \mathbb{C} & \longrightarrow & \mathbb{C} \times \mathbb{C}^* \\ (z_1, \check{w}_2) & \xrightarrow{\check{p}_r} & (z_1, w_2). \end{array}$$

where $\check{w}_2 = x_2 + 2\sqrt{-1}\check{y}_2$, $w_2 = x_2 + 2\sqrt{-1}y_2$, and $y_2 = p_r(\check{y}_2)$.

Since f is \mathbb{T}^2 -invariant function, i.e., f is independent of y , so f can be naturally treated as the function defined in the coordinate (z_1, \check{w}_2) . As in (5.1), let W be the determinant of the Hessian of f . Then

Lemma 7.3 *For any $u \in C^\infty(\mathfrak{h}^*, v_{\mathfrak{h}^*})$, consider the following affine transformation on u :*

$$\tilde{u}(\xi) = \lambda u(A^{-1}(\xi)) + \eta\xi_1 + b\xi_2 + c, \quad (7.4)$$

where $A(\xi_1, \xi_2) = (\alpha\xi_1, \beta\xi_2 + \gamma)$ and $\lambda = \alpha$. Then it induces an affine transformation in complex coordinate (z_1, \check{w}_2) :

$$B_{\mathbb{C}}(z_1, \check{w}_2) = (e^{\frac{\eta}{2}}z_1, \frac{\alpha}{\beta}\check{w}_2 + b). \quad (7.5)$$

Moreover,

- $\tilde{f}(z) = \alpha f(B_{\mathbb{C}}^{-1}z) + \gamma x_2 - \gamma b - c$;
- $\tilde{W}(z) = \beta^2 e^{-\eta} W(B_{\mathbb{C}}^{-1}z)$;
- $\tilde{\Psi}(z) = \alpha^{-1} \Psi(B_{\mathbb{C}}^{-1}z)$;
- $\tilde{\mathcal{K}}(z) = \alpha^{-1} \mathcal{K}(B_{\mathbb{C}}^{-1}z)$;
- $\|\nabla \log |\mathcal{S}(\tilde{f})|\|_{\tilde{f}}^2(z) = \alpha^{-1} \|\nabla \log |\mathcal{S}(f)|\|_f^2(B_{\mathbb{C}}^{-1}z)$,

where $z = (z_1, \check{w}_2)$.

Note that all these functions are \mathbb{T}^2 -invariant, hence the formulae can be push-forwarded from $\mathbb{C} \times \mathbb{C}$ down to $\mathbb{C} \times \mathbb{C}^*$. The lemma follows from a direct calculation. We omit it.

7.2 Uniform control of sections

In this section, we consider functions $u \in \mathcal{C}^\infty(\mathfrak{h}^*, v_{\mathfrak{h}^*}; K_o)$ (i.e., $u = \xi_1 \log \xi_1 + \xi_2^2 + \psi$ is strictly convex and $|\mathcal{S}(u)| \leq K_o$, where $\psi \in C^\infty(\mathfrak{h}^*)$) with the property

$$\Theta_u(p)d_u^2(p, \mathfrak{t}_2^*) \leq C_5. \quad (7.6)$$

Let $\mathfrak{z}^\circ \in \mathbb{U}_{\mathfrak{h}^*}$ be any point such that $d(\mathfrak{z}^\circ, Z) = 1$ and $\mathfrak{z}^* \in B_1(\mathfrak{z}^\circ) \cap Z$. Without loss of generality, we assume that \mathfrak{z}° is a representative point of its orbit and assume that it is on \mathfrak{t} (cf. Remark 1.3). Let p°, p^* be their images of moment map τ_f .

Remark 7.4 *In this section, when we consider a point $z \in \mathbb{U}_{\mathfrak{h}^*}$, without loss of generality, we assume z is the representative of its \mathbb{T}^2 -orbit (cf Remark 1.3). Hence when $p = \tau_f(z)$, we assume that $z = \nabla^u(p) \in \mathfrak{t}$. If z is on Z , we may assume that it is the point in \mathfrak{t}_2 . For such z we write $\hat{\tau}_f^{-1}(p)$.*

By adding a linear function we normalize u such that p° is the minimal point of u ; i.e.,

$$u(p^\circ) = \inf u. \quad (7.7)$$

Let \check{p} be the minimal point of u on \mathfrak{t}_2^* , the boundary of \mathfrak{h}^* . By adding some constant to u , we require that

$$u(\check{p}) = 0. \quad (7.8)$$

By a coordinate translation we can assume that

$$\xi(\check{p}) = 0. \quad (7.9)$$

(see Figure 1). Denote $z^* = \nabla^u(p^*)$. Let

$$S_0 := \left\{ (-\infty, x_2) \in \mathfrak{t}_2 \mid \left| \int_{x_2}^{x_2(z^*)} \sqrt{f_{22}} dx_2 \right| \leq 1 \right\}.$$

By a coordinate transformation

$$A(\xi_1, \xi_2) = (\xi_1, \beta \xi_2) \quad (7.10)$$

we can normalize u such that

$$|S_0| = 10. \quad (7.11)$$

In fact, (7.10) induces a coordinate transformation in (x_1, x_2) as following

$$A(x_1, x_2) = (x_1, \beta^{-1} x_2),$$

then by choosing the proper β we have (7.11). The above transformations are affine transformations in (7.4) with $\lambda = \alpha = 1$. It is easy to see that $\mathcal{K}, \Psi, \|\nabla \log |\mathcal{S}(f)|\|_f^2$ and $W(z)/W(z')$ for any $z, z' \in \mathbb{U}_{\mathfrak{h}^*}$ are invariant under these transformations.

We say (u, p°, \check{p}) is a *minimal-normalized-triple* if u satisfies (7.7), (7.8), (7.9), (7.11) and

$$d(p^\circ, \mathfrak{t}_2^*) = 1.$$

Note that \check{p} is determined by u and p° already.

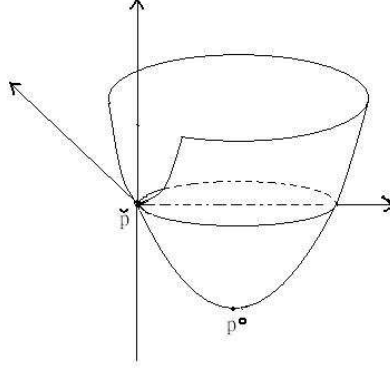


Figure 1:

Lemma 7.5 *Let (u, p°, \check{p}) be a minimal-normalized triple. Then there is a constant $C_1 > 1$ such that*

$$u(\check{p}) - u(p^\circ) \geq C_1^{-1}.$$

Proof. By (7.6) we have

$$\Theta \leq 4C_5, \quad \text{in } B_{\frac{1}{2}}(p^\circ).$$

Then by a similar argument of the proof of the claim in Theorem 2.9, we conclude that $u(p) - u(p^\circ) \geq \delta$ for any p not in $B_{\frac{1}{2}}(p^\circ)$. In particular, \check{p} is such a point. Hence, we prove the lemma. q.e.d.

Lemma 7.6 *Let $(u_k, p_k^\circ, \check{p}_k)$ be a sequence of minimal-normalized triples with*

$$\lim_{k \rightarrow \infty} \max |\mathcal{S}(u_k)| = 0,$$

Then there is a constant $C_1 > 1$ such that

$$C_1^{-1} \leq u_k(\check{p}_k) - u(p_k^\circ) \leq C_1$$

when k large enough.

Proof. The lower bound is proved in Lemma 7.5.

By adding a constant to u_k , we assume that $u_k(p_k^\circ) = 0$. Now suppose that $u_k(\check{p}_k)$ has no upper bound. Then we can choose a sequence of constants $N_k \rightarrow \infty$ such that

$$0 < N_k < u_k(\check{p}_k), \quad \lim_{k \rightarrow \infty} N_k \max |\mathcal{S}(u_k)| = 0.$$

For each u_k we take an affine transformation $\hat{A}_k := (A_k, (N_k)^{-1})$ to get a new function \tilde{u}_k , i.e.,

$$\tilde{u}_k = (N_k)^{-1} u_k \circ (A_k)^{-1}.$$

Then original section $S_{u_k}(p^\circ, N_k)$ is transformed to be $S_{\tilde{u}_k}(A_k p_k^\circ, 1)$. We choose A_k such that the latter one is normalized (cf. §3.1). Then, by Lemma 2.2,

$$\lim_{k \rightarrow \infty} \mathcal{S}(\tilde{u}_k) = \lim_{k \rightarrow \infty} N_k \mathcal{S}(u_k) = 0,$$

$$\lim_{k \rightarrow \infty} d_{\tilde{u}_k}(A_k p_k^\circ, A_k \mathfrak{t}_2^*) = \lim_{k \rightarrow \infty} \frac{1}{\sqrt{N_k}} d_{u_k}(p^\circ, \mathfrak{t}_2^*) = 0.$$

On the other hand, by Theorem 3.6 we conclude that \tilde{u}_k locally uniformly $C^{3,\alpha}$ -converges to a strictly convex function \tilde{u}_∞ in a neighborhood of p_∞° , which is the minimal point of u_∞ and the limit of $A_k p_k^\circ$. In particular, there is a constant $C_3 > 0$ such that

$$d_{\tilde{u}_k}(A_k p_k^\circ, A_k \mathfrak{t}_2^*) \geq C_3.$$

We get a contradiction. ■

Definition 7.7 *Let (u, p°, \check{p}) be a minimal-normalized-triple. Let $N = \max(200, 100bC_1, 100 \exp(100\sqrt{C_{10}}))$, where b is the constant in Proposition 3.9 and C_{10} is the constant in Lemma 6.5 with $N_0 = 4, n = 2$. If the following inequalities hold*

$$\mathcal{K}(z) \leq 4, \quad \forall z \in \tau_f^{-1}(B_N(p^\circ)); \quad (7.12)$$

$$\frac{1}{4} \leq \frac{W(z)}{W(z')} \leq 4, \quad \forall z, z' \in \tau_f^{-1}(B_N(p^\circ)), \quad (7.13)$$

we say that (u, p°, \check{p}) is a bounded-normalized triple.

Let u_k be a sequence given in Lemma 7.6 and $(u_k, p_k^\circ, \check{p}_k)$ be a sequence bounded-normalized triples. In this subsection, we mainly prove that the sections $S_{u_k}(p_k^\circ, \sigma_k)$ are L -normalized for some constant L that is independent of k , where $\sigma_k = \delta |\min u_k|$ and δ is the constant in Proposition 3.9. By Lemma 7.6, we already know that σ_k are uniformly bounded. By Lemma 3.4 we need only prove the following theorem.

Theorem 7.8 *Let $(u_k, p_k^\circ, \check{p}_k)$ be a sequence of bounded-normalized triples. Suppose that $\lim_{k \rightarrow \infty} |S(u_k)| = 0$. Let*

$$\Omega_k = S_{u_k}(p_k^\circ, \sigma_k).$$

Then there are constants c_{in} and c_{out} , independent of k , such that

$$c_{in} \leq wd_i(\Omega_k), \quad i = 1, 2.$$

and $|\xi_i(\Omega_k)| \leq c_{out}, i = 1, 2$.

The proof is contained in the following lemmas: Lemma 7.9, 7.10 and 7.13.

Lemma 7.9 *$wd_1(\Omega_k)$ has a uniformly lower bound.*

Proof. Let

$$E_k = \{p \in S_{u_k}(p_k^\circ, \sigma_k) \mid \xi_2(p) = \xi_2(p_k^\circ)\}$$

and p_k^\pm be its right and left ends. Then we claim that

Claim. There is a constant $C_2 > 0$ such that $|p_k^\circ - p_k^\pm| \geq C_2^{-1}$.

Proof of claim. We prove the claim for p^- . The proof for p^+ is identical.

If it is not true, then $|\xi_1(p_k^\circ) - \xi_1(p_k^-)| \rightarrow 0$. We omit the index k . Take an affine transformation on u

$$\tilde{u}(\xi) = u(A^{-1}\xi),$$

where A is the normalizing transformation of $S_u(p^\circ, \delta^{-1}\sigma)$. By Theorem 3.6 we conclude that \tilde{u}_k uniformly $C^{3,\alpha}$ -converges to a smooth and strictly convex function \tilde{u}_∞ in $S_{\tilde{u}_\infty}(\tilde{p}_\infty^\circ, \sigma)$. Note that the geodesic distance and the ratio $\det(u_{ij})(\xi)/\det(u_{ij})(p^\circ)$ are base-affine transformation invariants, we have for any $p \in S_u(p^\circ, \sigma)$

$$C_3^{-1} \leq d(p^\circ, \partial S_u(p^\circ, \sigma)) \leq C_3, \quad (7.14)$$

$$C_3^{-1} \leq \det(u_{ij})(p)/\det(u_{ij})(p^\circ) \leq C_3, \quad (7.15)$$

for some constant $C_3 > 1$ independent of k .

On the other hand, by the convexity of u , we have

$$|\partial_1 u(p^-)| \geq \frac{\sigma}{|p^\circ - p^-|} \geq \frac{\delta}{C_1 |p^\circ - p^-|} \rightarrow \infty. \quad (7.16)$$

By the coordinate corresponding $z_1 = e^{\frac{u_1}{2}}$, $z_2 = w_2$, we have

$$W(z) = \frac{1}{4} \det \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right) e^{-x_1}(z) = \frac{1}{4} \left[\det \left(\frac{\partial^2 u}{\partial \xi_i \partial \xi_j} \right) \right]^{-1} e^{-u_1}(\tau_f(z)).$$

Let $z^- \in \tau_f(p^-) \cap B_N(\mathfrak{z}^\circ)$. As $\partial_1 u(p^-) < 0$ and $\partial_1 u(p^\circ) = 0$, from (7.15) and (7.16) we conclude that

$$W(z^-)/W(z^\circ) = \exp(-\partial_1 u(p^-)) \frac{\det(u_{ij})(p^\circ)}{\det(u_{ij})(p^-)} \rightarrow +\infty,$$

which contradicts to the assumption (7.13). This completes the proof of the claim.

Since the width is at least $|p^+ - p^-|$, we get its lower bound. q.e.d.

Recall $\mathfrak{z}^* \in Z$ is the point such that $d(\mathfrak{z}^\circ, \mathfrak{z}^*) = 1$. Let $p^* = \tau_f(\mathfrak{z}^*)$.

Lemma 7.10 $wd_2(\Omega_k)$ has a uniform lower bound.

Proof. We divide the proof into three steps.

Step 1. There is a constant $C_4 = \exp(100\sqrt{C_{10}})$ such that for any $p \in B_{20}(p^\circ) \cap \mathfrak{t}_2^*$

$$u(p) - u(p^\circ) \leq C_4, \quad (7.17)$$

where C_{10} is the constant in Lemma 6.5 with $N_0 = 4, a = 100, n = 2$. In particular,

$$u(p^*) - u(p^\circ) \leq C_4. \quad (7.18)$$

Proof of Step 1. Let p^\bullet be the point such that

$$u(p^\bullet) = \max_{B_{20}(p^\circ) \cap \mathfrak{t}_2^*} u.$$

We only need to prove $u(p^\bullet)$ satisfies (7.17). By a coordinate translation $\tilde{\xi}_1 = \xi_1, \tilde{\xi}_2 = \xi_2 - \alpha$ we can assume that $\xi(p^\bullet) = (0, 0)$. Obviously $u(p^\bullet) - u(p^\circ)$ is invariant under the coordinate transformation.

Let $\mathfrak{z}^\circ \in \tau_f^{-1}(p^\circ)$ and $\mathfrak{z}^\bullet \in \tau_f^{-1}(p^\bullet)$ with $d(\mathfrak{z}^\circ, \mathfrak{z}^\bullet) \leq 20$. Let f be the function associated with u (cf. section 1.5). Obviously $f(\mathfrak{z}^\bullet) = \inf_{\mathbb{C} \times \mathbb{C}^*} f$. Note that $u = \xi_1 \log \xi_1 + \psi$ with $\psi \in C^\infty(\mathfrak{h}^*)$. Following from

$$\lim_{\xi \rightarrow p^\bullet} \xi_1 (\log \xi_1 + \psi_1) = 0, \quad \xi(p^\bullet) = 0, \quad \frac{\partial u}{\partial \xi_1}(p^\circ) = \frac{\partial u}{\partial \xi_2}(p^\circ) = 0$$

we have

$$\sum \frac{\partial u}{\partial \xi_i} \xi_i(p^\circ) = 0, \quad \lim_{\xi \rightarrow p^\bullet} \sum \frac{\partial u}{\partial \xi_i} \xi_i = 0. \quad (7.19)$$

Then

$$f(\mathfrak{z}^\bullet) + u(p^\bullet) = 0, \quad f(\mathfrak{z}^\circ) + u(p^\circ) = 0.$$

It follows that $|u(p^\circ) - u(p^\bullet)| = |f(\mathfrak{z}^\circ) - f(\mathfrak{z}^\bullet)|$. Applying Lemma 6.5 to f in $B_{100}(\mathfrak{z}^\bullet)$ with $N_0 = 4, n = 2$, we have

$$|f(\mathfrak{z}^\circ) - f(\mathfrak{z}^\bullet)| \leq C_4$$

for $C_4 = \exp(100\sqrt{C_{10}})$. Then the claim follows.

Step 2. Let $q_1 = (-\infty, m_1)$, $q_2 = (-\infty, m_2)$ be two ends of S_0 , $m_1 < m_2$.

Denote $p_i = (0, \frac{\partial f}{\partial x_2}(m_i))$. We have

$$\begin{aligned} 1 &\leq \left| \int_{m_i}^{x_2(z^*)} \sqrt{f_{22}} dx_2 \right| = \left| \int_{\xi_2(p_i)}^{\xi_2(p^*)} \sqrt{u_{22}} d\xi_2 \right| \\ &\leq |\xi_2(p_i) - \xi_2(p^*)| \left| \int_{\nabla f(S_0)} u_{22} d\xi_2 \right| \\ &\leq 10 |\xi_2(p_i) - \xi_2(p^*)|. \end{aligned}$$

It follows that

$$|\xi_2(p_i) - \xi_2(p^*)| \geq \frac{1}{10}. \quad (7.20)$$

Step 3. For $C_5 = C_4 + 1$, by the result of Step 1 and the definition of S_0 , we have

$$\nabla^u(S_0) \subset B_{20}(p^\circ) \cap \mathfrak{t}_2^* \subset S_u(p^\circ, C_5) \cap \mathfrak{t}_2^*.$$

Let p_3, p_4 be the boundary of $S_u(p^\circ, C_5) \cap \mathfrak{t}_2^*$. Then by the result of Step 2 we have

$$|p_3 - p_4| \geq |p_1 - p_2| \geq \frac{1}{5}. \quad (7.21)$$

Take the triangle $\triangle p^\circ p_3 p_4 \subset \mathfrak{t}^*$. On the other hand, let

$$P^\circ = (p^\circ, u(p^\circ)), P_3 = (p_3, u(p_3)), P_4 = (p_4, u(p_4))$$

They form a triangle $\triangle P^\circ P_3 P_4$ in 3-dimensional space $\mathfrak{t}^* \times \mathbb{R}$. It is above the graph of u over $\triangle p^\circ p_3 p_4$.

Now we cut the triangle $\triangle P^\circ P_3 P_4$ by the horizontal plane $\xi_3 = u(p^\circ) + \sigma$, i.e:

$$\triangle^\sigma = \{(\xi_1, \xi_2, \xi_3) \in \triangle P^\circ P_3 P_4 \mid \xi_3 \leq u(p^\circ) + \sigma\}.$$

Let \triangle_σ be the projection of \triangle^σ onto \mathfrak{t}^* . It is easy to see that $\triangle_\sigma \subset S_u(p^\circ, \sigma)$. By the similarity between triangles \triangle^σ and $\triangle P^\circ P_3 P_4$, we conclude that the upper edge has length at least $\sigma(10C_5)^{-1}$. It follows that

$$wd_2(S_u(p^\circ, \sigma)) \geq wd_2(\triangle_\sigma) \geq \sigma(10C_5)^{-1}, \quad q.e.d.$$

From the first two steps in the proof, we have the following corollary.

Corollary 7.11 *There is a constant $C_6 > 0$, depending only on C_{10} , such that*

$$|u_2(p^*) - u_2(p^\circ)| = |u_2(p^*)| \leq C_6.$$

Proof. Without loss of generality we assume that $\xi_2(p_1) < \xi_2(p^*) < \xi_2(p_2)$. By the convexity of u we conclude that

$$\frac{u(p_1) - u(p^*)}{\xi_2(p_1) - \xi_2(p^*)} \leq u_2(p^*) \leq \frac{u(p_2) - u(p^*)}{\xi_2(p_2) - \xi_2(p^*)}.$$

Then Corollary follows from (7.17) and (7.20). Thus,

$$-10C_4 \leq u_2(p^*) \leq 10C_4.$$

Back to the proof of Theorem 7.8, we have

Lemma 7.12 *The Euclid volume of $\Omega_k = S_{u_k}(p_k^\circ, \sigma_k)$ has a uniform upper bound.*

Proof. First we claim that there is a constant $C_9 < N$ such that

$$\tau_f^{-1}(\Omega_k) \subset B_{C_9}(\mathfrak{z}^\circ).$$

Note that $B_{C_9}(\mathfrak{z}^\circ) \subset \mathbb{C} \times \mathbb{C}^*$ with the coordinate system (z_1, w_2) . We omit k again. Note that for points in $\nabla^u(\Omega)$, this is true. It remains to consider the

distance for points on the fiber over points in $\nabla^u(\Omega)$. Our discuss is taken place on affine-log coordinate chart.

Suppose that

$$\mathfrak{z}^\circ = (x_1^\circ, x_2^\circ, 0, 0).$$

On the one hand, by the definition of S_0 and (7.11) we can find a point $z^\alpha = (-\infty, x_2^\alpha, 0, 0) \in B_2(\mathfrak{z}^\circ)$ with $(-\infty, x_2^\alpha) \in S_0$ such that

$$f_{22}(z^\alpha) \leq \frac{1}{10}. \quad (7.22)$$

This says that the metric along the torus of y_2 -direction over z^α is bounded. In fact, If this is not true, we have $f_{22}|_{S_0} > \frac{1}{10}$. By a direct calculation we have

$$\int_{S_0} \sqrt{f_{22}} dx_2 > \int_{S_0} \frac{1}{4} dx_2 = \frac{5}{2}.$$

It contradicts to (7.11). On the other hand, let $p^\alpha = \tau_f(z^\alpha)$, then $p^\alpha \in \mathfrak{t}_2^*$ and

$$\lim_{\xi \rightarrow p^\alpha} u^{11} = \lim_{\xi \rightarrow p^\alpha} \frac{u_{22}}{u_{11}u_{22} - u_{12}^2} = \lim_{\xi \rightarrow p^\alpha} \frac{\xi_1 u_{22}}{(1 + \xi_1 \psi_{11})u_{22} - \xi_1 u_{12}^2} = 0.$$

Hence

$$f_{11}(z^\alpha) = 0. \quad (7.23)$$

This says that the metric along the torus of y_1 -direction over z^α is bounded.

Now consider any point

$$\tilde{z} = (x_1, x_2, y_1, y_2)$$

over $z = (x_1, x_2, 0, 0) \in \nabla^u(\Omega)$. We construct a bounded path from \tilde{z} to \mathfrak{z}° as the following:

- (i) keeping y coordinates, move "parallelly" from \tilde{z} to the point $(-\infty, x_2^\alpha, y_1, y_2)$ over z^α , denote $\hat{z}^\alpha = (-\infty, x_2^\alpha, y_1, y_2)$;
- (ii) move along the fiber over z^α along y_1 -direction from \hat{z}^α to the point $(-\infty, x_2^\alpha, 0, y_2)$, denote $\tilde{z}^\alpha = (-\infty, x_2^\alpha, 0, y_2)$;
- (iii) move along the fiber over z^α along y_2 -direction from \tilde{z}^α to z^α ;
- (iv) keeping y coordinates, move from z^α to \mathfrak{z}° .

By a direct calculation we have

$$\begin{aligned} d(\tilde{z}, \mathfrak{z}^\circ) &\leq d(\tilde{z}, \hat{z}^\alpha) + d(\hat{z}^\alpha, \tilde{z}^\alpha) + d(\tilde{z}^\alpha, z^\alpha) + d(z^\alpha, \mathfrak{z}^\circ) \\ &\leq 2 + C_1 b + 4\pi + 2 := C_9. \end{aligned}$$

where we used (7.22) and (7.23) in the second inequality. The claim is proved.

Since the Ricci curvature tensor is bounded, by the Bishop Volume Comparison Theorem we have

$$Vol(\tau_f^{-1}(\Omega)) \leq Vol(B_{C_9}(\mathfrak{z}^\circ)) \leq C_{10}$$

for some constant $C_{10} > 0$. By a direct calculation we have

$$\begin{aligned}
4\pi^2 Vol_E(\Omega) &= 4\pi^2 \int_{\Omega} d\xi_1 d\xi_2 \\
&= \int_{\mathbb{T}^2} \int_{\nabla^u(\Omega)} \det \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right) dx_1 dx_2 dy_1 dy_2 \\
&= Vol(\tau_f^{-1}(\Omega)) \leq C_{10},
\end{aligned}$$

where $Vol_E(\Omega)$ denotes the Euclidean volume of the set Ω . q.e.d.

We have the following consequence,

Lemma 7.13 *There is a Euclidean ball $D_R(0)$ such that for any k*

$$S_{u_k}(p_k^\circ, C_5) \subset D_R(0).$$

In particular,

$$\Omega_k \subset D_R(0)$$

and

$$|\xi|(p_k^*) \leq R, \quad |\xi|(p_k^\circ) \leq R. \quad (7.24)$$

Proof. Since $wd_i(\Omega_k)$ is bounded below, let c_{in} be the bound. Let

$$\tilde{\Omega}_k = S_{u_k}(p_k^\circ, |u_k(p_k^\circ)| + C_5).$$

Then by the convexity of u_k and the similarity we have

$$\begin{aligned}
Vol_E(\tilde{\Omega}_k) &\leq \left(\frac{|u_k(p_k^\circ)| + C_5}{\sigma_k} \right)^2 Vol_E(\Omega_k) \\
&\leq \frac{(1 + C_1 C_5)^2 Vol_E(\Omega_k)}{\delta^2} \leq \frac{(1 + C_1 C_5)^2 C_{10}}{\pi^2 \delta^2}.
\end{aligned}$$

Note that $\check{p} = (0, 0) \in \tilde{\Omega}_k$ and

$$wd_i(\tilde{\Omega}_k) \geq wd_i(\Omega_k) \geq c_{in}.$$

Then there is a constant $R > 0$ independent of k such that for any $p \in \tilde{\Omega}_k$ $|\xi_i(p)| \leq \frac{R}{2}$, for $i = 1, 2$. q.e.d.

Lemma 7.14 *There is a constant $C_{11} > 0$ independent of k , such that for any $p \in B_8(\check{z}^\circ)$,*

$$|z_1|(p) + |w_2|(p) \leq C_{11}.$$

Proof. Note that

$$|z_1|^2 = e^{\frac{w_1 + \bar{w}_1}{2}} = e^{x_1}, \quad |w_2|^2 \leq |x_2|^2 + 4\pi^2.$$

We only need prove that x_1 and $|x_2|$ are uniformly bounded above. Set

$$\tilde{f} = f - \frac{\partial f}{\partial x_2}(q_1)(x_2 - x_2(q_1)) - f(q_1), \quad \hat{f} = f - \frac{\partial f}{\partial x_2}(z^*)(x_2 - x_2(z^*)) - f(z^*).$$

Then

$$\tilde{f}(q_1) = \inf \tilde{f} = 0, \quad \hat{f}(z^*) = \inf \hat{f} = 0.$$

Using Lemma 6.5 we have

$$|\tilde{f}(q) - \tilde{f}(q_1)| \leq N_1, \quad |\hat{f}(q) - \hat{f}(z^*)| \leq N_1. \quad (7.25)$$

for any $q \in B_8(\mathfrak{z}^o)$, where $N_1 = \exp\{20\sqrt{C_{10}}\}$. Note that

$$\tilde{f} = \hat{f} + (\xi_2(p^*) - \xi_2(p_1))x_2 + d_0,$$

where $d_0 = \xi_2(p_1)x_2(q_1) - f(q_1) - \xi_2(p^*)x_2(z^*) + f(z^*)$. Then by (7.25) we have

$$\begin{aligned} 2N_1 &\geq |\tilde{f}(q) - \tilde{f}(q_1)| + |\tilde{f}(z^*) - \tilde{f}(q_1)| \\ &\geq |\tilde{f}(q) - \tilde{f}(z^*)| \\ &\geq |\xi_2(p^*) - \xi_2(p_1)||x_2(z^*) - x_2(q)| - |\hat{f}(q) - \hat{f}(z^*)| \\ &\geq \frac{1}{10}|x_2(z^*) - x_2(q)| - N_1 \end{aligned} \quad (7.26)$$

where we use (7.20) and (7.25) in the last inequality. It follows from (7.26) and Corollary 7.11 that

$$|x_2(q)| \leq |x_2(z^*)| + 30N_1 \leq C_6 + 30N_1 = N_2 \quad (7.27)$$

for any $q \in B_8(\mathfrak{z}^o)$.

Let $q \in B_8(p^o)$ be a point with $u_1(q) = \max_{B_8(p^o)} u_1$. Then

$$x_1(\nabla^u(q)) = \max_{p \in B_8(\nabla^u(p^o))} x_1(p).$$

Set $\tilde{u} = u - u_1(q)\xi_1 - u_2(q)\xi_2$. Then $\tilde{u}(q) = \inf \tilde{u}$. As in the Step 1 of Lemma 7.10, we have $\tilde{u}(p^*) - \tilde{u}(q) \leq C_4$. Since $\tilde{u}(q) = \inf \tilde{u}$, we conclude that

$$\begin{aligned} C_4 &\geq \tilde{u}(p^*) - \tilde{u}(p^o) \\ &= (u(p^*) - u(p^o)) + u_1(q)\xi_1(p^o) - u_2(q)(\xi_2(p^*) - \xi_2(p^o)) \\ &\geq u_1(q)\xi_1(p^o) - N_2|\xi_2(p^*) - \xi_2(p^o)|. \end{aligned} \quad (7.28)$$

where we use (7.27) and $u(p^o) = \inf u$ in the last inequality. By Lemma 7.13 and (7.24) we have

$$|\xi_2(p^*)| \leq R, \quad |\xi_2(p^o)| \leq R, \quad |\xi_2(p^*) - \xi_2(p^o)| \leq 2R. \quad (7.29)$$

Substituting (7.29) into (7.28) and using $\xi_1(p^o) \geq C_2^{-1} > 0$ we have

$$x_1(\nabla^u(q)) \leq C_{12}$$

for some constant $C_{12} > 0$ independent of k . Hence on $B_8(\mathfrak{z}^o)$, $|z|$ is uniformly bounded above. q.e.d.

7.3 A convergence theorem

We will use Theorem 7.8 to prove a convergence theorem.

Theorem 7.15 *Let $(u_k, p_k^\circ, \check{p}_k)$ be a sequence of bounded-normalized triples. Suppose that $\lim_{k \rightarrow 0} |S(u_k)| \rightarrow 0$. Then*

- (1). u_k locally uniformly $C^{3,\alpha}$ -converges to a strictly convex function u_∞ in $B_{\frac{N}{8}}(p_k^\circ) \cap (\mathfrak{h}^* \setminus \mathfrak{t}_2^*)$; the similar conclusion can be stated in $\mathfrak{U}_{\mathfrak{h}^*} \setminus Z$.
- (2). \mathfrak{z}_k^* converges to a point \mathfrak{z}_∞^* and f_k uniformly $C^{3,\alpha}$ -converges to a function f_∞ in $D_{a_1}(\mathfrak{z}_\infty^*)$. Here a_1 is the constant in Theorem 6.10.

Proof. Applying Theorem 7.8 and Lemma 3.4, we conclude that $\Omega_k := S_{u_k}(p_k^\circ, \sigma_k)$ are L -normalized, where L is independent of k . Then by Theorem 3.6 and Remark 3.7 (1) is proved.

By Lemma 7.14 and $B_7(\mathfrak{z}^*) \subset B_8(\mathfrak{z}^\circ)$ we have, in $B_7(\mathfrak{z}^*)$,

$$|z| \leq C_{11}, \quad \mathcal{K}(z) \leq 4, \quad C^{-1} \leq W(z) \leq C \quad (7.30)$$

for some constant $C > 0$ independent of k . Here we use the fact that $W(\mathfrak{z}^\circ)$ is bounded due to (1). By adding a linear function we can assume that $f(\mathfrak{z}^*) = \inf_{\mathbb{C} \times \mathbb{C}^*} f = 0$.

Note that the geodesic distance and (7.30) are invariant under the transformation of adding a linear function. We get f satisfies the assumption in Theorem 6.10. By Theorem 6.10 with $C_1 = C_{11} + 4 + C, a = 4$, we conclude that f_k uniformly $C^{3,\alpha}$ -converges to a function f_∞ in $D_{a_1}(\mathfrak{z}_\infty^*)$ for some constant $a_1 > 0$ independent of k . ■

7.4 Proof of Theorem 7.1

Put

$$\mathcal{W}_f = \frac{\min_{B_a(\mathfrak{z}_o) \cap Z_\ell} W}{\max_{B_a(\mathfrak{z}_o)} W}, \quad \mathcal{R}_f = (\mathcal{K}(f) + \|\nabla \log |S(f)|\|^2).$$

Suppose that the theorem is not true, then there is a sequence of functions f_k and a sequence of points $\mathfrak{z}'_k \in B_{a/2}(\mathfrak{z}_o)$ such that

$$\mathcal{W}_k \mathcal{R}_k(\mathfrak{z}'_k) a^2 \rightarrow \infty \quad \text{as } k \rightarrow \infty, \quad (7.31)$$

where $\mathcal{W}_k := \mathcal{W}_{f_k}, \mathcal{R}_k = \mathcal{R}_{f_k}$. Note that $\mathcal{W}_k \leq 1$. We apply the argument in Remark 4.1 for the function

$$F_k(z) := \mathcal{W}_k \mathcal{R}_k(z) [d_{f_k}(z, \partial B_a(\mathfrak{z}_o))]^2$$

defined in $B_a(\mathfrak{z}_o)$, the geodesic ball with respect to the metric ω_{f_k} . Suppose that it attains its maximum at z_k° . By (7.31) we have

$$\lim_{k \rightarrow \infty} F_k(z_k^\circ) \rightarrow +\infty. \quad (7.32)$$

Put

$$d_k = \frac{1}{2}d_{f_k}(z_k^o, \partial B_a(\mathfrak{z}_o)).$$

Then in $B_{d_k}(z_k^o)$

$$\mathcal{R}_k \leq 4\mathcal{R}_k(z_k^o).$$

Let $z_k^* \in Z$ be the point such that $d(z_k^o, z_k^*) = d(z_k^o, Z)$. Then

Lemma 7.16 $z_k^* \in B_{d_k}(z_k^o)$.

Proof. Since $\mathcal{R}_k(z_k^o)d_k^2 \rightarrow \infty$ and $\mathcal{R}_k(z_k^o)d^2(z_k^o, Z) \leq C_5$, (cf. Corollary 4.5 and Corollary 4.6) we conclude that $z_k^* \in B_{d_k}(z_k^o)$. q.e.d.

Let q_k^o, q_k^*, \dots be the points of the image of z_k^o, z_k^*, \dots under the moment map τ_f .

Now we perform the affine blowing-up analysis to derive a contradiction (cf. section 7.1). We take an affine transformation as in Lemma 7.3 by setting

$$\alpha_k = \beta_k^2 = \mathcal{R}(z_k^o), \quad \eta_k = \log \alpha_k + \log W(z_k^*),$$

and $b_k = c_k = \gamma_k = 0$. Then we assume that the notations are changed as $u_k \rightarrow \tilde{u}_k, f_k \rightarrow \tilde{f}_k, d_k \rightarrow \tilde{d}_k$ and etc.

Claim:

1. $\alpha_k \rightarrow \infty, \tilde{d}_k \rightarrow \infty$ as $k \rightarrow \infty$;
2. $\tilde{\mathcal{R}}_k(\tilde{z}_k^o) = 1$ and $\tilde{\mathcal{R}}_k \leq 4\tilde{\mathcal{R}}_k(z_k^o) = 4$ in $B_{\tilde{d}_k}(\tilde{z}_k^o)$;
3. $\lim_{k \rightarrow \infty} \max_{B_{\tilde{d}_k}(\tilde{z}_k^o)} |\tilde{S}_k| = 0$,
4. $\tilde{W}(\tilde{z}_k^*) = 1$;
5. $\lim_{k \rightarrow \infty} \max_{B_{\tilde{d}_k}(\tilde{z}_k^o)} \|\nabla \tilde{S}_k\|_{\tilde{f}} = 0$;
6. $\tilde{W}_k^{1/2} \tilde{\Psi}_k \rightarrow 0$ in $B_{\tilde{d}_k/2}(\tilde{z}_k^o)$.

Proof of claim. (1-4) follows from Lemma 7.3. For (5) we use the fact $|\nabla \tilde{S}| = |\nabla \log \tilde{S}| |\tilde{S}|$ when $\tilde{S} \neq 0$. Now we prove (6). By Lemma 6.3, in $B_{\tilde{d}_k/2}(\tilde{z}_k^o)$ we have

$$\tilde{W}_k^{1/2} \tilde{\Psi}_k \leq C \left(\max_{B_{\tilde{d}_k}(\tilde{z}_k^o)} \tilde{W}_k \right)^{1/2} \left[\max_{B_{\tilde{d}_k}(\tilde{z}_k^o)} \left(|\tilde{S}_k| + \|\nabla \tilde{S}_k\|^{2/3} \right) + \tilde{d}_k^{-1} + \tilde{d}_k^2 \right].$$

Then we get

$$\left(\max_{B_{\tilde{d}_k}(\tilde{z}_k^o)} \tilde{W}_k \right)^{1/2} \max |\tilde{S}_k| \leq (\mathcal{W}_k)^{-1/2} \mathcal{R}(z_k^o)^{-1} \max |S_k| \leq (\mathcal{W}_k \mathcal{R}(z_k^o))^{-1} \max |S_k| \rightarrow 0;$$

$$(\max_{B_{\tilde{d}_k}(\tilde{z}_k^o)} \tilde{W}_k)^{1/2} \max \|\nabla \tilde{S}_k\|^{\frac{2}{3}} \leq (\mathcal{W}_k)^{-\frac{1}{2}} \mathcal{R}(z_k^o)^{-\frac{2}{3}} \max |S_k|^{\frac{2}{3}} \leq C(\mathcal{W}\mathcal{R}(z_k^o))^{-\frac{2}{3}} \rightarrow 0;$$

$$(\max_{B_{\tilde{d}_k}(\tilde{z}_k^o)} \tilde{W}_k)^{1/2} \tilde{d}_k^{-1} \leq (\mathcal{W}_k \mathcal{R}(\tilde{z}^o) d_k^2)^{-1/2} \rightarrow 0$$

and so is $(\max_{B_{\tilde{d}_k}(\tilde{z}_k^o)} \tilde{W}_k)^{1/2} \tilde{d}_k^{-2}$. Hence (6) is verified. q.e.d.

In particular, by (6), we know that for any fixed R , then for any small constant $\epsilon > 0$ when k large

$$1 - \epsilon \leq \tilde{W}_k(z) \leq 1 + \epsilon, \forall z \in B_R(z_k^o). \quad (7.33)$$

To derive a contradiction we need the convergency of f_k . We omit again the index k . By Lemma 7.17 below, we can find a point \mathfrak{z}^o in $B_2(\tilde{z}^*)$ such that $d(\mathfrak{z}^o, Z) = c$. For simplicity, without loss of generality, we assume that $c = 1$ and $d(\mathfrak{z}^o, \tilde{z}^*) = 1$. Let $p^o = \tau_f(\mathfrak{z}^o)$. To apply Theorem 7.15, we need following preparations:

- by affine transformations in (7.4) with $\lambda = \alpha = 1$, we can minimal-normalize $(\tilde{u}, \tilde{p}^o, \tilde{p})$ (cf. section 7.2),
- since $\tilde{\mathcal{R}}, \tilde{\mathcal{K}}, \tilde{\Psi}$ and $\tilde{W}(z)/\tilde{W}(z')$ for any $z, z' \in B_N(\tilde{z}^o)$ are invariant under these transformations, the bounded condition is certainly true by the claim and (7.33).

After this transformation, we assume that the notations are changed as $\tilde{u} \rightarrow \bar{u}$, $\tilde{d} \rightarrow \bar{d}$ and etc. In this way we get a sequence $(\bar{u}_k, \bar{p}_k^o, \bar{p}_k)$, that satisfies the conditions in Theorem 7.15. We discuss two cases.

Case 1. There is a constant $C' > 0$ such that $\bar{d}(\bar{z}_k^o, \bar{z}_k^*) \geq C'$. Then by $\mathcal{R}_k(z_k^o) d^2(z_k^o, Z) \leq C_5$ and $\mathcal{R}_k(z_k^o) = 1$ we have

$$C' \leq \bar{d}(\bar{z}_k^o, \bar{z}_k^*) = \bar{d}(\bar{z}_k^o, Z) \leq \sqrt{C_5}.$$

Applying (1) of Theorem 7.15 we conclude that \bar{z}_k^o converges to a point \bar{z}_∞^o and \bar{u}_k uniformly $C^{3,\alpha}$ -converges to a function \bar{u}_∞ in a neighborhood of $\tau_f(\bar{z}_\infty^o)$.

Case 2. $\lim_{k \rightarrow \infty} \bar{d}(\bar{z}_k^o, \bar{z}_k^*) = 0$. Applying (2) of Theorem 7.15 we conclude that \bar{z}_k^* converges to a point \bar{z}_∞^* and \bar{f}_k locally uniformly $C^{3,\alpha}$ -converges to a function \bar{f}_∞ in the ball $D_{a_1}(\bar{z}_\infty^*)$. Then \bar{z}_k^o converges to a point \bar{z}_∞^o and \bar{f}_k locally uniformly $C^{3,\alpha}$ -converges to a function \bar{f}_∞ in the ball $D_{a_1}(\bar{z}_\infty^o)$.

Hence for both cases we have \bar{z}_k^o converges to a point \bar{z}_∞^o and \bar{f}_k locally uniformly $C^{3,\alpha}$ -converges to a function \bar{f}_∞ in a neighborhood of \bar{z}_∞^o , and

$$\bar{W} \equiv \text{const.}, \quad \bar{\Psi} \equiv 0, \quad C_1^{-1} \leq \bar{f}_{ij} \leq C_1, \quad (7.34)$$

in a neighborhood of \bar{z}_∞^o , where $C_1 > 0$ is a constant. Here (7.34) follows from (7.33) and the $C^{3,\alpha}$ -convergence of \bar{f}_k .

Claim: $\lim_{k \rightarrow \infty} \max_{B_{b_1}(\bar{z}_\infty^*)} \|\nabla \log |\mathcal{S}(\bar{f}_k)|\|_{\bar{f}_k}^2 = 0.$

Proof of claim. Suppose the affine transformations from u to \bar{u} are (we omit the index k)

$$\bar{\xi}_1 = \bar{\alpha}\xi_1, \quad \bar{\xi}_2 = \bar{\beta}(\xi_2 + \gamma), \quad \bar{u}(\bar{\xi}_1, \bar{\xi}_2) = \bar{\alpha}u((\bar{\alpha})^{-1}\bar{\xi}_1, (\bar{\beta})^{-1}(\bar{\xi}_2 - \gamma)) - l(\bar{\xi}),$$

where γ is a constant and $l(\bar{\xi})$ is a linear function. Obviously,

$$\lim_{k \rightarrow \infty} \bar{\alpha}_k = +\infty. \quad (7.35)$$

From the $C^{3,\alpha}$ -convergence of \bar{f}_k , we have in the ball $B_{b_1}(\bar{z}_\infty^*)$

$$|z_1| + |w_2| \leq C'_1, \quad (C'_1)^{-1} \leq \bar{f}_{i\bar{j}} \leq C'_1 \quad (7.36)$$

for some constant $C'_1 > 1$. Then in $\mathfrak{t} \cap B_{b_1}(\bar{z}^*)$, in terms of coordinates \bar{x}_1, \bar{x}_2 ,

$$C_2^{-1} \leq \frac{\partial^2 \bar{f}}{\partial \bar{x}_2^2} \leq C_2, \quad \left| \frac{\partial^2 \bar{f}}{\partial \bar{x}_i \partial \bar{x}_j} \right| \leq C_2. \quad (7.37)$$

It follows that

$$C_2^{-1} \leq \frac{\partial^2 \bar{u}}{\partial \bar{\xi}_2^2} \leq C_2, \quad |\bar{u}^{ij}| \leq C_2 \quad (7.38)$$

for some constant $C_2 > 0$ independent of k , where (\bar{u}^{ij}) is the inverse matrix of the matrix $\left(\frac{\partial^2 \bar{u}}{\partial \bar{\xi}_i \partial \bar{\xi}_j} \right)$. By $\frac{\partial^2 \bar{u}}{\partial \bar{\xi}_2^2} = \frac{\bar{\alpha}}{\bar{\beta}^2} \frac{\partial^2 u}{\partial \xi_2^2}$ and $\frac{\partial^2 h}{\partial \xi_2^2}|_{\ell \cap \tau_f(B_a(\mathfrak{z}_o))} \geq \mathbf{N}_5^{-1}$ we conclude that

$$\frac{\bar{\beta}^2}{\bar{\alpha}} \geq \mathbf{N}_5^{-1} C_2^{-1}.$$

It follows from (7.35) that

$$\lim_{k \rightarrow \infty} \bar{\beta}_k = +\infty. \quad (7.39)$$

Since

$$\|\nabla \log |\mathcal{S}(\bar{f})|\|_{\bar{f}}^2 = \|\nabla \log |\mathcal{S}(f)|\|_f^2 = \sum \bar{u}^{ij} \frac{\partial \xi_k}{\partial \bar{\xi}_i} \frac{\partial \xi_l}{\partial \bar{\xi}_j} \frac{\partial \log |\mathcal{S}(f)|}{\partial \xi_k} \frac{\partial \log |\mathcal{S}(f)|}{\partial \xi_l},$$

and

$$\frac{\partial \xi_1}{\partial \bar{\xi}_1} = \frac{1}{\bar{\alpha}}, \quad \frac{\partial \xi_2}{\partial \bar{\xi}_2} = \frac{1}{\bar{\beta}}, \quad \frac{\partial \xi_1}{\partial \bar{\xi}_2} = \frac{\partial \xi_2}{\partial \bar{\xi}_1} = 0,$$

by (7.35), (7.38) and (7.39) the claim is proved.

Combing this claim and $\tilde{\mathcal{R}}_k(\bar{z}_k^o) = \bar{\mathcal{R}}_k(\bar{z}_k^o)$ we get, for k large enough,

$$\bar{\mathcal{K}}_k(\bar{z}_k^o) \geq 1 - \epsilon. \quad (7.40)$$

Note that

$$\frac{\partial^{i+j} \mathcal{S}(\bar{u})}{\partial \bar{\xi}_1^i \partial \bar{\xi}_2^j} = (\bar{\alpha})^{-i-1} (\bar{\beta})^{-j} \frac{\partial^{i+j} \mathcal{S}(u)}{\partial \xi_1^i \partial \xi_2^j}.$$

Then by (7.35), (7.39) and $\|\mathcal{S}(u_k)\|_{C^3(\Delta)} \leq \mathbf{N}_5$ we have

$$\lim_{k \rightarrow \infty} \|\mathcal{S}(\bar{u}_k)\|_{C^3} = 0,$$

where $\|\cdot\|_{C^3}$ denotes the Euclidean C^3 -norm in $(\bar{\xi}_1, \bar{\xi}_2)$. Then, by the regularity theorem, \bar{f}_k $C^{6,\alpha}$ -converges to \bar{f}_∞ in a neighborhood of \bar{z}_∞^o . Then by (7.34) we have $\bar{\mathcal{K}} \equiv 0$. This contradicts to (7.40).

The theorem is proved. \blacksquare

Lemma 7.17 *Let $z^* \in Z$. Suppose that in $B_2(z^*)$, $\mathcal{K} \leq 4$. Then there is a constant $c > 0$, independent of f , such that there is a point z^o in $B_1(z^*)$ satisfying*

$$d(z^o, B_2(z^*) \cap Z) = c.$$

Obviously $c \leq 1$. Hence $d(z^o, Z) = c$.

Proof. Without loss of generality, we assume that the w -coordinate of z^* is

$$x_1 = -\infty, y_1 = 0, x_2 = 0, y_2 = 0.$$

Recall that \mathfrak{t} is identified with $\mathfrak{t} \times 2\sqrt{-1}\{1\}$ of $\mathbf{U}_{\mathfrak{h}^*}$. Then when we consider $B_2(z^*)$, we restrict ourself on $B_2(z^*) \cap \mathfrak{t}$. Similarly, when we consider Z we treat it as the line $\mathfrak{t}_2 = \{x_1 = -\infty\}$, which is the dual to \mathfrak{t}_2^* (when treat Z as a 1-dimensional toric manifold).

If the claim of the lemma does not hold, then there exists a sequence of function f_k such that: for any point $z \in \partial B_1^{(k)}(z^*)$

$$d(z, B_2^{(k)}(z^*) \cap Z) \leq \frac{1}{k}.$$

Let $\hat{z}_k \in B_2^{(k)}(z^*) \cap Z$ such that

$$d_{f_k}(z, \hat{z}_k) = d_{f_k}(z, B_2^{(k)}(z^*) \cap Z) \leq \frac{1}{k}.$$

Then

$$1 - \frac{1}{k} \leq d_{f_k}(z^*, \hat{z}_k) \leq 1 + \frac{1}{k}.$$

Let $A_k = B_1^{(k)}(z^*) \cap \mathfrak{t}_2$ and split it into A_k^\pm according the sign of x_2 -coordinate. Define

$$\mathcal{A}_k^\pm = \{z \in \partial B_1(z^*) | d(z, (A_k^\pm)) \leq 1/k\}.$$

Then \mathcal{A}_k^\pm are non-empty and are close subsets of a $\partial B_1^{(k)}(z^*)$. Since they cover $\partial B_1^{(k)}(z^*)$, their intersection is non-empty. Choose a point $z_k^o \in \mathcal{A}_k^+ \cap \mathcal{A}_k^-$. Choose two points $z_k^\pm \in A_k^\pm$ such that $d(z_k^o, z_k^\pm) \leq 1/k$. We normalize the function f_k by adding a linear function (cf. Section 7.1) such that it achieves its minimum at z_k^\pm . We denote the function by f_k^\pm respectively.

Then we have following facts:

Fact 1: $|f_k^\pm(z^+) - f_k^\pm(z^-)| \leq Ck^{-1}$ for some constant C . In fact, by (6.8) and the assumption, we have

$$|\log(1 + f_k^\pm)(z_k^+) - \log(1 + f_k^\pm)(z_k^-)| \leq \sqrt{C_{10}}d(z_k^+, z_k^-).$$

It follows that

$$|f_k^\pm(z_k^+) - f_k^\pm(z_k^-)| \leq C'd(z_k^+, z_k^-) \leq Ck^{-1}.$$

Fact 2: let $\bar{f}_k^\pm = f_k^\pm|_Z$ and \bar{d} denote the geodesic distance on \mathfrak{t}_2 with respect to $G_{\bar{f}_k^\pm}$. Then $\bar{d}(z_k^+, z_k^-) \geq 1$ when k large. In fact,

$$\bar{d}(z_k^\pm, z^*) \geq d(z_k^\pm, z^*) \geq 1 - 1/k.$$

We omit the index k for simplicity. Now we focus on \bar{f}^\pm . By changing coordinate on x_2 we may assume that

$$x_2(z^+) - x_2(z^-) = 1.$$

Set $x_2^\pm = x_2(z^\pm)$. Then

$$1 \leq \bar{d}^2(z^+, z^-) \leq \left(\int_{x_2^-}^{x_2^+} \sqrt{\bar{f}_{22}^\pm} dx_2 \right)^2 \leq \int_{x_2^-}^{x_2^+} \bar{f}_{22}^\pm dx_2 \leq |\bar{f}_2^\pm(z^+) - \bar{f}_2^\pm(z^-)|.$$

We summarize that we have two convex functions \bar{f}^\pm on the *unit* interval $[x_2^-, x_2^+]$ that are different up to a linear function and they have the following properties:

- $|\nabla \bar{f}^+(x_2^+) - \nabla \bar{f}^+(x_2^-)| \geq 1, |\nabla \bar{f}^-(x_2^+) - \nabla \bar{f}^-(x_2^-)| \geq 1;$
- $|\bar{f}^+(x_2^+) - \bar{f}^+(x_2^-)| \rightarrow 0, |\bar{f}^-(x_2^+) - \bar{f}^-(x_2^-)| \rightarrow 0, \text{ as } k \rightarrow \infty;$
- $\nabla \bar{f}^+(x_2^+) = 0, \nabla \bar{f}^-(x_2^-) = 0.$

This is impossible as k large. We get a contradiction. q.e.d.

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